

Fluctuations, stability and instability of a distributed particle filter with local exchange

Kari Heine^{a,1,*}, Nick Whiteley^a

^a*School of Mathematics, University Walk, Bristol, BS8 1TW, United Kingdom*

Abstract

We study a distributed particle filter proposed by Bolić et al. (2005). This algorithm involves m groups of M particles, with interaction between groups occurring through a “local exchange” mechanism. We establish a central limit theorem in the regime where M is fixed and $m \rightarrow \infty$. A formula we obtain for the asymptotic variance can be interpreted in terms of colliding Markov chains, enabling analytic and numerical evaluations of how the asymptotic variance behaves over time, with comparison to a benchmark algorithm consisting of m independent particle filters. We prove that subject to regularity conditions, when m is fixed both algorithms converge time-uniformly at rate $M^{-1/2}$. Through use of our asymptotic variance formula we give counter-examples satisfying the same regularity conditions to show that when M is fixed neither algorithm, in general, converges time-uniformly at rate $m^{-1/2}$.

Keywords: hidden Markov model, particle filter, central limit theorem, asymptotic variance, local exchange, sequential Monte Carlo

2010 MSC: 60F05, 60F99, 60G35

*Corresponding author

Email addresses: `k.heine@ucl.ac.uk` (Kari Heine), `Nick.Whiteley@bristol.ac.uk` (Nick Whiteley)

¹Present address: Department of Statistical Science, University College London, Gower Street, London, WC1E 6BT, United Kingdom

1. Introduction

Since their introduction in [1], particle filters have become very popular tools in engineering, signal processing, econometrics and various other disciplines for approximate nonlinear filtering of hidden Markov models (HMM's). Investigations of particle filters have generated book-length studies, notably [2], demonstrating the well-developed state of knowledge about convergence rates, fluctuations, propagation of chaos, large deviations and various other properties, with more recent contributions to the literature focussing on specific algorithmic mechanisms, such as adaptive resampling [3, 4].

Trends in the development of computers towards distributed and parallel architectures have influenced particle filtering methodology. One of the main bottlenecks for computational efficiency when implementing particle filters is the interaction between particles which occurs in the resampling step. This step is important because it ensures that the algorithm exhibits certain time-uniform convergence properties, but is difficult to parallelize.

A significant piece of work from the engineering literature which addresses this difficulty is [5], introducing an algorithm we refer to as the Local Exchange Particle Filter (LEPF), in which groups of particles are spread across computational units. What makes this algorithm unusual is that the m groups of M weighted particles interact through an “exchange” mechanism, which places it outside the frameworks of many existing studies, notably [2, 3, 4]. The practical rationale for the LEPF is to achieve a compromise between communication efficiency of the algorithm and the benefits brought about by resampling. In particular the interaction between particles in the LEPF occurs in a localized manner, making it suited to implementation on a network of computing devices without the need for global connections.

Despite substantial interest in [5] from practitioners—it has 250 citations according to Google scholar at the time of writing—relatively little is known about convergence properties of LEPF. Indeed the question of whether it truly exhibits the same time-uniform convergence properties as the original particle

filter of [1] has not been fully answered. The few papers on analysis of the LEPF appear to be [6, 7] and the recent technical report [8]. [6] concerns analysis over a single time-step, and [7, 8] provide proofs of time-uniform convergence of the particle filtering approximation error, in L_1 and L_p norms respectively, in the regime where M is fixed and $m \rightarrow \infty$, for an algorithm of which the LEPF as we present it is a special case. However, the proofs of [7, 8] rely on key hypotheses on the particle weights which they do not rigorously verify, and which seem difficult to check in general. The results of [7, 8] also do not establish a particular rate of convergence.

The structure of this paper and outline of our main contributions are as follows (precise statements are given later). In Section 2 we introduce the setup of the filtering problem, present the LEPF and describe the main result of [7, 8]. We also introduce a standard algorithm consisting of m independent bootstrap particle filters (IBPF), each with M particles. The independence in the IBPF makes it very easy to parallelize, so from a computational point of view it is a natural alternative to the LEPF. In this paper the convergence properties of the IBPF, which are already well-understood, serve as benchmarks against which to compare the LEPF.

Section 3 introduces a general algorithm of which the LEPF and IBPF are special cases, and gives our main result, Theorem 1, a central limit theorem (CLT) for the error in particle approximation of prediction filter distributions, in the regime where M is fixed and $m \rightarrow \infty$. We address time-uniform convergence in Section 4. Our first result here is a positive one: that under strong but standard regularity conditions, in L_p norm the error from the LEPF converges time-uniformly with rate $M^{-1/2}$, in the regime where m is fixed and $M \rightarrow \infty$. The same is true of the IBPF. Our second result, Proposition 5 in Section 4.2, shows that growth without bound of the asymptotic variance in our CLT is sufficient to rule out time-uniform convergence at rate $m^{-1/2}$ in the regime where M is fixed and $m \rightarrow \infty$.

Section 5 investigates various properties of the asymptotic variance for the LEPF and compares them to those of the IBPF. In particular, we show by

examples in Sections 5.1 and 5.2 that under conditions which can be considered very favourable for performance, the asymptotic variance for the LEPF and IBPF can grow over time without bound. This can be considered a negative result for the LEPF, since the sequence of asymptotic variances (over time) for the original particle filter of [1] has been shown under weaker conditions to be bounded, or tight when the observations in the HMM are treated as random [9, 10, 11, 12, 13]. Moreover, combined with Proposition 5 in Section 4.2, these examples serve as counter-examples to time-uniform convergence at rate $m^{-1/2}$. This does not contradict the time-uniform convergence results of [7, 8], since the latter results do not pertain to a specific convergence rate, and they concern the updated filtering distributions. However, our Proposition 5 allows us to confirm that a hypothesis slightly stronger than that of [8] does not hold in general, even under favourable conditions. Section 6 contains further discussion and interpretation of our results. Our analysis allows us to explain qualitatively why the asymptotic variance for the LEPF may be lower or grow over time more slowly than that for the IBPF, and we illustrate this phenomenon with numerical results.

Some clarifications about originality are in order. To the knowledge of the authors, our CLT is the first result of its kind for the LEPF. Our starting point to prove this result consists of a martingale decomposition and error bounds, Proposition 1 in Section 3, which is an application of a result obtained by the authors in [14] for a class of algorithms which includes the LEPF. However, we emphasise that Proposition 1 is only one of the first steps towards the CLT itself, leaving us with substantial work to do. In our study of time-uniform convergence, we also appeal to a result of [14] (Proposition 4 in the present paper), but again we have some work to do in dealing with the specifics of the LEPF. We also point out that despite some superficial similarities, the details of LEPF and our analysis differ substantially from those of some resampling algorithms studied recently by the authors in [15].

Notation

For any measurable space $(\mathbb{X}, \mathcal{X})$ we use $\mathcal{M}(\mathbb{X})$, $\mathcal{P}(\mathbb{X})$ and $\mathcal{B}(\mathbb{X})$ to denote the set of measures, probability measures and the set of bounded and measurable functions defined on \mathbb{X} , respectively. \mathbb{N} includes 0. For any \mathbb{N} -valued $m \geq 1$ we write $[m] := \{1, \dots, m\}$. Whenever summation over a single variable appears without the summation set made explicit, the sum is taken over the set $[N]$, i.e. $\sum_i \equiv \sum_{i=1}^N$ and for summations over multiple variables we write $\sum_{(i_1, \dots, i_p)} \equiv \sum_{i_1} \cdots \sum_{i_p}$. We use Id to denote the identity mapping for any domain of definition and $\mathbf{1}$ to denote a constant function equal to 1 everywhere. For any function $\varphi : A \rightarrow \mathbb{R}$, we define $\varphi^{\otimes 2}(x, y) := \varphi(x)\varphi(y)$ for all $x, y \in A$. For $\varphi \in \mathcal{B}(\mathbb{X})$ we define $\|\varphi\|_\infty := \sup_x |\varphi(x)|$ and $\text{osc}(\varphi) := \sup_{x, y} |\varphi(x) - \varphi(y)|$. For any $\mu, \nu \in \mathcal{M}(\mathbb{X})$, $\mu \otimes \nu$ denotes the product measure and $\mu^{\otimes 2} := \mu \otimes \mu$. We use δ_x to denote the point mass located at x . We define $\lfloor x \rfloor := \max(z \in \mathbb{Z} : z \leq x)$ and $(y \bmod x) := y - \lfloor (y-1)/x \rfloor x$. All random variables we encounter are considered to be defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with expectation w.r.t. \mathbb{P} denoted by \mathbb{E} . Convergence in probability under \mathbb{P} is denoted by $\xrightarrow{\mathbb{P}}$.

2. Filtering framework and the LEPF

Let $X = (X_n)_{n \in \mathbb{N}}$ be a Markov chain taking values in a measurable Polish space $(\mathbb{X}, \mathcal{X})$, having initial distribution $\pi_0 \in \mathcal{P}(\mathbb{X})$ and transition kernel $f : \mathbb{X} \times \mathcal{X} \rightarrow [0, 1]$,

$$X_0 \sim \pi_0, \quad X_n \sim f(X_{n-1}, \cdot), \quad \forall n \geq 1.$$

Let $Y = (Y_n)_{n \in \mathbb{N}}$ be a process taking values in a measurable Polish space $(\mathbb{Y}, \mathcal{Y})$ such that $(Y_n)_{n \in \mathbb{N}}$ are conditionally independent given X , with the conditional distribution of Y_n given X being

$$Y_n \sim g(X_n, \cdot), \quad \forall n \in \mathbb{N},$$

for a probability kernel $g : \mathbb{X} \times \mathcal{Y} \rightarrow [0, 1]$. For all $x \in \mathbb{X}$, we assume $g(x, \cdot)$ admits a density with respect to a σ -finite measure on $(\mathbb{Y}, \mathcal{Y})$, and the same

notation $g(x, \cdot)$ will be used for denoting this density. From here on, we consider a fixed \mathbb{Y} -valued observation sequence $(y_n)_{n \in \mathbb{N}}$, write $g_n(x) := g(x, y_n)$ for all $x \in \mathbb{X}$, and assume that the following mild regularity condition holds.

Assumption 1. *For all $n \in \mathbb{N}$, $g_n \in \mathcal{B}(\mathbb{X})$ and $g_n(x) > 0$ for all $x \in \mathbb{X}$.*

We focus on approximating the $\mathcal{P}(\mathbb{X})$ -valued prediction filter sequence $(\pi_n)_{n \in \mathbb{N}}$, which cannot be computed exactly, except in some special cases. This sequence is defined for all $n \geq 1$, by the recursion $\pi_n = \Phi_n(\pi_{n-1})$ where $\Phi_n : \mathcal{P}(\mathbb{X}) \rightarrow \mathcal{P}(\mathbb{X})$ is the operator

$$\Phi_n(\mu)(A) := \frac{\int_{\mathbb{X}} g(x, y_{n-1}) f(x, A) \mu(dx)}{\int_{\mathbb{X}} g(x, y_{n-1}) \mu(dx)}, \quad \forall A \in \mathcal{X}, \mu \in \mathcal{P}(\mathbb{X}).$$

If $(y_n)_{n \in \mathbb{N}}$ is replaced by the random sequence $(Y_n)_{n \in \mathbb{N}}$, then π_n is a version of the conditional distribution of X_n given Y_0, \dots, Y_{n-1} .

The algorithm which is our main object of study is one of several proposed in [5] and there called the ‘‘Distributed Resampling with Non-proportional Allocation and Local Exchange’’ algorithm. For brevity, we refer to it as the LEPF. It is shown in Algorithm 1. At each time step n , this algorithm delivers a collection of $N = Mm$ particles $\zeta_n = \{\zeta_n^i : i \in [N]\}$ and weights $\{W_n^i : i \in [N]\}$, and the weighted empirical measure

$$\pi_n^N := \frac{\sum_i W_n^i \delta_{\zeta_n^i}}{\sum_i W_n^i}, \quad (1)$$

is regarded as an approximation to π_n . The sampling steps of Algorithm 1 should be understood to mean that the particles $\zeta_n = \{\zeta_n^i : i \in [N]\}$ are conditionally independent given $\zeta_0, \dots, \zeta_{n-1}$. Within each of the m groups of equal size M , the particles are drawn according to a common resampling/proposal mechanism. Indeed one can read off from Algorithm 1 that

$$W_n^i = W_n^j \quad \text{and} \quad \mathbb{P}(\zeta_n^i \in \cdot \mid \zeta_0, \dots, \zeta_{n-1}) = \mathbb{P}(\zeta_n^j \in \cdot \mid \zeta_0, \dots, \zeta_{n-1}), \quad \forall i, j \in G_k, \quad (2)$$

and the parameter $\theta \in \{1, \dots, M-1\}$ influences the interaction between groups via the indices L^i .

Algorithm 1 Local exchange particle filter

for $i = 1, \dots, Mm$

Set $W_0^i = 1$ and sample $\zeta_0^i \sim \pi_0$

Set $L^i = (i + \theta) \bmod Mm$

for $k = 1, \dots, m$

Set $G_k = \{(k-1)M + 1, \dots, (k-1)M + M\}$

for $n = 1, 2, \dots$

for $k = 1, \dots, m$

for $i \in G_k$

Set $W_n^i = (Mm)^{-1} \sum_{j \in G_k} W_{n-1}^{L^j} g_{n-1}(\zeta_{n-1}^{L^j})$

Sample $\zeta_n^i \mid \zeta_0, \dots, \zeta_{n-1} \sim \frac{\sum_{j \in G_k} W_{n-1}^{L^j} g_{n-1}(\zeta_{n-1}^{L^j}) f(\zeta_{n-1}^{L^j}, \cdot)}{\sum_{j \in G_k} W_{n-1}^{L^j} g_{n-1}(\zeta_{n-1}^{L^j})}$

In this paper, we primarily focus on the asymptotic regime M fixed, $m \rightarrow \infty$. Interest in this regime stems from parallel and distributed implementations: typically the sampling and weight computations for the m groups are performed concurrently by a network of m computers, so the regime M fixed, $m \rightarrow \infty$ can be thought of as corresponding to an increasingly large network, in which each computer handles M particles, see [5] for details.

[7, 8] studied an algorithm of which the LEPF as we present it in Algorithm 1 is a special case. Our mapping $i \mapsto L^i$ is a particular instance of the mapping denoted by β in [7, 8] and if one sets their exchange period parameter $n_0 = 1$, one recovers Algorithm 1. The generality of β in [7, 8] allows for other patterns of interaction between particles, beyond the ones considered in the present article. Whilst we focus on the prediction filter distributions π_n , [7, 8] focus on particle approximations of the updated filtering distributions $\hat{\pi}_n(A) := \pi_n(g_n \mathbb{I}_A) / \pi_n(g_n)$, $A \in \mathcal{X}$, $n \geq 0$. To allow us to state their result, for each $n \geq 0$ let $\{\hat{\zeta}_n^i; i \in [N]\}$ be random variables which are conditionally independent given ζ_0, \dots, ζ_n , with

$$\mathbb{P}(\hat{\zeta}_n^i \in \cdot \mid \zeta_0, \dots, \zeta_n) = \frac{\sum_{j \in G_k} W_n^{L^j} g_n(\zeta_n^{L^j}) \delta_{\zeta_n^{L^j}}(\cdot)}{\sum_{j \in G_k} W_n^{L^j} g_n(\zeta_n^{L^j})}, \quad \forall i \in G_k,$$

and

$$\hat{\pi}_n^N := \frac{\sum_i W_{n+1}^i \delta_{\hat{\zeta}_n^i}}{\sum_i W_{n+1}^i}.$$

The $\{\hat{\zeta}_n^i; i \in [N]\}$ can be understood as “integrated out” in Algorithm 1.

In the notation of the present paper and with $\widetilde{W}_n^k := W_n^i = W_n^j$ for all $i, j \in G_k$ and $k \in [m]$, the key hypothesis of [8, Assumption 3] can be equivalently written as follows: there exist $\epsilon \in [0, 1)$ and $q \geq 4$ such that,

$$\sup_{m \geq 1} \sup_{n \geq 0} m^{q-\epsilon} \mathbb{E} \left[\left| \max_{k \in [m]} \frac{\widetilde{W}_n^k}{\sum_{j \in [m]} \widetilde{W}_n^j} \right|^q \right] < \infty. \quad (3)$$

Under this hypothesis, plus additional but standard regularity conditions, the main result of [8] is: for any $\varphi \in \mathcal{B}(\mathbb{X})$, $M \geq 1$ and $1 \leq p \leq q$ with q as in (3),

$$\lim_{m \rightarrow \infty} \sup_{n \geq 0} \mathbb{E}[|\hat{\pi}_n^{Mm}(\varphi) - \hat{\pi}_n(\varphi)|^p]^{1/p} = 0.$$

A similar result for the case $p = 1$ was established in [7] under stronger conditions. However, in [8] the hypothesis (3) is not rigorously verified, and only empirical evidence that it holds is presented. We shall comment further on (3) in Section 4.2.

The role of the indices L^i in the LEPF is made more transparent if one compares to an alternative algorithm, what we term *independent bootstrap particle filters* (IBPF), shown in Algorithm 2 below. The IBPF amounts to m independent copies of the original bootstrap particle filter of [1], each with $M = N/m$ particles. Indeed one can read off from Algorithm 2 that for the IBPF the m collections of particles $\{\zeta_n^i : i \in G_k, n \in \mathbb{N}\}$, $k \in [m]$ are independent, making the IBPF very easy to parallelise and hence in practice it is a natural alternative to the LEPF. Algorithm 2 also clearly satisfies (2), and one could write the “Sample” step more simply as:

$$\mathbb{P}(\zeta_n^i \in \cdot | \zeta_0, \dots, \zeta_{n-1}) = \frac{\sum_{j \in G_k} g_{n-1}(\zeta_{n-1}^j) f(\zeta_{n-1}^j, \cdot)}{\sum_{j \in G_k} g_{n-1}(\zeta_{n-1}^j)}, \quad \forall i \in G_k,$$

but the presentation of Algorithm 2 highlights the connection the LEPF: if in Algorithm 1 one were to set $\theta = 0$, so $L^i = i$, then one recovers exactly Algorithm 2. With the weights W_n^i as calculated in the IBPF, one again regards π_n^N as in

(1) as an approximation to π_n , and the statistical independence between groups means that convergence properties of the IBPF in the regime where M is fixed and $m \rightarrow \infty$ are relatively easy to study.

Algorithm 2 Independent bootstrap particle filters

for $i = 1, \dots, Mm$

Set $W_0^i = 1$ and sample $\zeta_0^i \sim \pi_0$

for $k = 1, \dots, m$

Set $G_k = \{(k-1)M + 1, \dots, (k-1)M + M\}$

for $n = 1, 2, \dots$

for $k = 1, \dots, m$

for $i \in G_k$

Set $W_n^i = (Mm)^{-1} \sum_{j \in G_k} W_{n-1}^j g_{n-1}(\zeta_{n-1}^j)$

Sample $\zeta_n^i \mid \zeta_0, \dots, \zeta_{n-1} \sim \frac{\sum_{j \in G_k} W_{n-1}^j g_{n-1}(\zeta_{n-1}^j) f(\zeta_{n-1}^j, \cdot)}{\sum_{j \in G_k} W_{n-1}^j g_{n-1}(\zeta_{n-1}^j)}$

3. Central limit theorem

3.1. A general algorithm and statement of the main result

The starting point for our analysis is to write down Algorithm 3 of which the LEPF and IBPF are special cases. We do this not just for the sake of generality. Instead Algorithm 3 affords us some notational simplifications and, more crucially, it allows us make clear that the LEPF is a special case of the so-called α SMC algorithm, introduced by the authors in [14]. In turn this later allows us to leverage some results of [14]—in particular Proposition 1 below—providing some building blocks for our CLT. The IBPF is also an instance of Algorithm 3 and this fact eases our presentation of comparisons between it and the LEPF in Section 5.

From henceforth, the integer $M \geq 1$ is, unless stated otherwise, assumed to be fixed. In Algorithm 3, α is a row-stochastic matrix, of size $N \times N$, with $N = Mm$. Assumption 2 introduces hypotheses on the matrix α for each value

Algorithm 3

for $i = 1, \dots, N$

Set $W_0^i = 1$ and sample $\zeta_0^i \sim \pi_0$

for $n = 1, 2, \dots$

for $i = 1, \dots, N$

Set $W_n^i = \sum_j \alpha^{ij} W_{n-1}^j g_{n-1}(\zeta_{n-1}^j)$

Sample $\zeta_n^i \mid \zeta_0, \dots, \zeta_{n-1} \sim (W_n^i)^{-1} \sum_j \alpha^{ij} W_{n-1}^j g_{n-1}(\zeta_{n-1}^j) f(\zeta_{n-1}^j, \cdot)$

$N \in \{Mm : m \geq 1\}$. To state these hypotheses precisely, we need to be clear about dependence of α on N and hence write α_N up until the end of Section 3.1, beyond which we revert to α to reduce notational clutter.

Assumption 2. For all $N \in \{Mm : m \geq 1\}$,

(2.1) α_N is doubly stochastic,

(2.2) for all $i, j \in [N]$ and $z \in \mathbb{Z}$,

$$\alpha_N^{ij} = \alpha_N^{(i+zM) \bmod N, (j+zM) \bmod N}.$$

Additionally, for some integer $\beta \geq 1$,

(2.3) $\alpha_N^{ij} = 0$ for $N \geq 2\beta + 1$ and $i, j \in [N]$ such that

$$\Delta(i, j) := \min_{\ell \in \mathbb{Z}} |i - j + \ell N| > \beta,$$

(2.4) there exists $\{\alpha_\infty^{ij} : i, j \in \mathbb{Z}\}$ such that for $N \geq 2\beta + 1$,

$$\alpha_\infty^{ij} = \alpha_N^{i \bmod N, j \bmod N} \mathbb{I}[|i - j| \leq \beta], \quad i, j \in \mathbb{Z}. \quad (4)$$

Assumption (2.1) allows us to apply results from [14] to Algorithm 3. Assumption (2.2) asserts that the elements on each diagonal of α_N are periodic with cycle length M . Intuitively, this captures the idea that the N particles in Algorithm 3 are in some sense organised into groups of size M . It is easily verified that the function Δ appearing in Assumption (2.3) is a metric on $[N]$,

in particular it is the graph distance on a cycle graph with vertex set $[N]$ where there is an edge between each $i \in [N]$ and $(i + 1) \bmod N$. Assumption (2.3) then asserts that α_N is a band matrix in the sense that elements further than β away from the main diagonal in metric Δ are equal to zero, in turn influencing the conditional independence structure of the particles in Algorithm 3. Finally Assumption (2.4) can be interpreted as meaning that there is some common structure to the matrices α_N as N grows, and loosely speaking, this common structure is captured in the “limiting” doubly infinite matrix α_∞ , which will show up later in our CLT.

Let us now state how the LEPF and IBPF fit in this framework. Consider

$$\begin{aligned}\alpha_N^{ij} &= M^{-1} \mathbb{I}[\lfloor (i-1)/M \rfloor = \lfloor ((j-\theta) \bmod N - 1)/M \rfloor] & \forall i, j \in [N] \\ \alpha_\infty^{ij} &= M^{-1} \mathbb{I}[\lfloor (i-1)/M \rfloor = \lfloor (j-\theta-1)/M \rfloor] & \forall i, j \in \mathbb{Z}.\end{aligned}\tag{5}$$

It is a matter of elementary but tedious manipulations to show that with $\alpha = \alpha_N$ as in (5), Algorithm 3 reduces to the LEPF as in Algorithm 1, and to check that Assumptions (2.1)–(2.3) hold with $\beta = M - 1 + \theta$. Checking Assumption (2.4) involves some less trivial work and a proof is provided in the Appendix.

To recover the IBPF from Algorithm 3, we take

$$\begin{aligned}\alpha_N^{ij} &= M^{-1} \mathbb{I}[\lfloor (i-1)/M \rfloor = \lfloor (j-1)/M \rfloor] & \forall i, j \in [N] \\ \alpha_\infty^{ij} &= M^{-1} \mathbb{I}[\lfloor (i-1)/M \rfloor = \lfloor (j-1)/M \rfloor] & \forall i, j \in \mathbb{Z}.\end{aligned}\tag{6}$$

With $\beta = M - 1$, checking Assumptions (2.1)–(2.3) is again elementary, and in this case Assumption (2.4) is obviously satisfied.

Figures 1a and 1b show the matrices defined in (5)–(6) in the case $N = 9$, $M = 3$ and $\theta = 1$. It follows from Assumptions (2.3) and (2.4) that α_∞ is, like each α_N , a row-stochastic matrix, which can be thought of as specifying the transition probabilities of a \mathbb{Z} -valued Markov chain. It turns out that the asymptotic variance in our CLT is expressed in terms of two copies of this chain. To this end, denote by $\mathbb{E}_{u,v}$, where $u, v \in \mathbb{Z}$, the expectation w.r.t. the law of the bi-variate backward Markov chain $(I_k, J_k)_{0 \leq k \leq n}$, where

$$\begin{aligned}(I_n, J_n) &\sim \delta_u \otimes \delta_v, \\ \mathbb{P}(I_k = i_k, J_k = j_k \mid I_{k+1} = i_{k+1}, J_{k+1} = j_{k+1}) &= \alpha_\infty^{i_{k+1}i_k} \alpha_\infty^{j_{k+1}j_k}.\end{aligned}\tag{7}$$

$$\begin{pmatrix}
0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\
1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\
1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\
1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3
\end{pmatrix}
\begin{pmatrix}
1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3
\end{pmatrix}$$

(a)
(b)

Figure 1: Matrices in (a) and (b) correspond to the LEPF and IBPF, respectively.

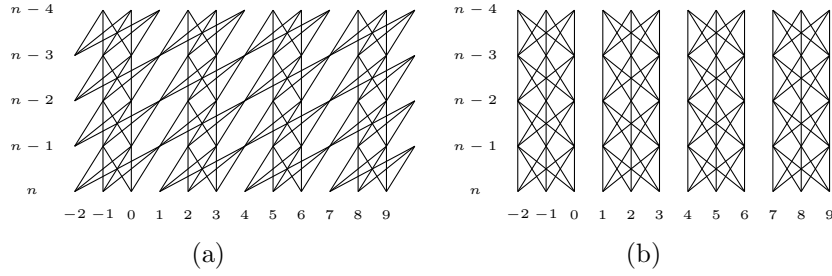


Figure 2: Some of the paths assigned positive probability by α_∞ for the (a) LEPF and (b) IBPF. In both cases $M = 3$ and in (a) $\theta = 1$.

Figure 2 illustrates some segments of paths for I (or J) which have strictly positive probability under the transitions α_∞ for the LEPF and IBPF, with $M = 3$ and $\theta = 1$.

Before stating our main result we introduce some more notation. For all $n \geq 1$, define non-negative kernels $Q_n : \mathbb{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ as

$$Q_n(x, A) := g_{n-1}(x)f(x, A), \quad \forall x \in \mathbb{X}, A \in \mathcal{X}, \quad (8)$$

and the corresponding operators on functions and measures

$$\begin{aligned} Q_n(\varphi)(x) &:= \int Q_n(x, dx') \varphi(x'), \quad \forall x \in \mathbb{X}, \varphi \in \mathcal{B}(\mathbb{X}), \\ \mu Q_n(A) &:= \int Q_n(x, A) \mu(dx), \quad \forall \mu \in \mathcal{P}(\mathbb{X}), A \in \mathcal{X}, \end{aligned}$$

respectively. Moreover we define for $n \geq 1$ and $0 \leq p < n$

$$\begin{aligned} Q_{p,p} &:= Id, & Q_{p,n} &:= Q_{p+1} \cdots Q_n, \\ \overline{Q}_{p,p} &:= Id, & \overline{Q}_{p,n} &:= \overline{Q}_{p+1} \cdots \overline{Q}_n, \end{aligned}$$

where $\overline{Q}_n := Q_n / \pi_{n-1}(g_{n-1})$ for all $n \geq 1$. Also let

$$\gamma_n := \pi_0 Q_{0,n}, \quad \forall n \geq 0.$$

Define the tensor-product kernel $Q_n^{\otimes 2}(x, y, d(x', y')) := Q_n(x, dx') Q_n(y, dy')$, with the corresponding operators on functions and measures written similarly to those for Q_n , and finally define operators \mathcal{C}_0 and \mathcal{C}_1 , such that for any $\varphi \in \mathcal{B}(\mathbb{X}^2)$,

$$\mathcal{C}_0(\varphi)(x, y) = \varphi(x, y) \quad \text{and} \quad \mathcal{C}_1(\varphi)(x, y) = \varphi(x, x), \quad \forall x, y \in \mathbb{X}.$$

We then have:

Theorem 1. *Fix $M > 1$ and $\beta > 0$ and suppose that Assumption 2 holds. Then for any $\varphi \in \mathcal{B}(\mathbb{X})$, Algorithm 3 has the property*

$$\sqrt{N}(\pi_n^N(\varphi) - \pi_n(\varphi)) \xrightarrow[N \rightarrow \infty]{d} \mathcal{N}(0, \sigma_n^2), \quad \forall n \in \mathbb{N},$$

where N goes to infinity along the sequence $\{Mm : m = 1, 2, \dots\}$, the following variances are assumed strictly positive,

$$\begin{aligned} \sigma_0^2 &= \pi_0((\varphi - \pi_0(\varphi))^2) \\ \sigma_n^2 &= \frac{1}{\gamma_n(\mathbf{1})^2 M} \sum_{\substack{0 \leq u < M \\ |v| \leq 2n\beta}} \mathbb{E}_{u, u+v} \left[\pi_0^{\otimes 2} \mathcal{C}_{\epsilon_0} Q_1^{\otimes 2} \mathcal{C}_{\epsilon_1} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n} (\overline{\varphi}^{\otimes 2}) \right], \quad n \geq 1, \end{aligned}$$

with $\epsilon_k = \mathbb{I}[I_k = J_k]$, for all $0 \leq k \leq n$, and $\overline{\varphi} := \varphi - \pi_n(\varphi)$.

Remark 1. Since the LEPF and IBPF are special cases of Algorithm 3, Theorem 1 applies to them immediately. Note that the only distinction between the asymptotic variances for LEPF and the IBPF arises from α_∞ , as given for these two algorithms in (5) and (6). In Section 5 we shall examine σ_n^2 for the LEPF and the IBPF in detail, which involves study of the I, J processes for these two algorithms.

3.2. Martingale array and the proof of the main result

Defining the random measures

$$\Gamma_n^N := \frac{1}{N} \sum_{i=1}^N W_n^i \delta_{\zeta_n^i}, \quad \bar{\Gamma}_n^N := \frac{\Gamma_n^N}{\gamma_n(\mathbf{1})}, \quad \forall n \in \mathbb{N}, \quad (9)$$

allows us to decompose the particle approximation error as

$$\sqrt{N}(\pi_n^N(\varphi) - \pi_n(\varphi)) = \sqrt{N}\bar{\Gamma}_n^N(\bar{\varphi}) - \pi_n^N(\bar{\varphi}) \frac{\sqrt{N}}{\gamma_n(\mathbf{1})}(\Gamma_n^N(\mathbf{1}) - \gamma_n(\mathbf{1})), \quad (10)$$

where $\bar{\varphi} := \varphi - \pi_n(\varphi)$.

Our overall strategy in proving Theorem 1 is to establish asymptotic normality of $\sqrt{N}\bar{\Gamma}_n^N(\bar{\varphi})$ as $N \rightarrow \infty$ using the CLT for martingale arrays [16], and to apply results from [14] to show that the second term on the r.h.s. of (10) converges to zero in probability. Our first step is to identify a martingale representation for $\sqrt{N}\bar{\Gamma}_n^N(\bar{\varphi})$, for which the setup is as follows.

Fix $n \in \mathbb{N}$ and $M \geq 1$. For given $m \geq 1$ and $\varphi \in \mathcal{B}(\mathbb{X})$ define, for $\varrho \in [Mm]$,

$$\xi_\varrho^m := \frac{1}{\sqrt{Mm}} \left(\bar{Q}_{0,n}(\varphi)(\zeta_0^i) - \pi_0 \bar{Q}_{0,n}(\varphi) \right), \quad (11)$$

and for $\varrho \in [(n+1)Mm] \setminus [Mm]$,

$$\xi_\varrho^m := \frac{W_p^i}{\sqrt{Mm}\gamma_p(\mathbf{1})} \left(\bar{Q}_{p,n}(\bar{\varphi})(\zeta_p^i) - \frac{\sum_j \alpha^{ij} W_{p-1}^j \bar{Q}_{p-1,n}(\bar{\varphi})(\zeta_{p-1}^j)}{\sum_j \alpha^{ij} W_{p-1}^j \bar{Q}_p(\mathbf{1})(\zeta_{p-1}^j)} \right), \quad (12)$$

where $p = p_m(\varrho)$, $i = i_m(\varrho)$ and

$$p_m(\varrho) := \left\lfloor \frac{\varrho - 1}{Mm} \right\rfloor \quad \text{and} \quad i_m(\varrho) := \varrho \bmod Mm.$$

Writing out the expression for W_p^i , $p \geq 1$, in Algorithm 3, using the fact that α is row-stochastic and Assumption 1,

$$W_p^{i_p} = \sum_{(i_0, \dots, i_{p-1})} \prod_{q=0}^{p-1} \alpha^{i_{q+1} i_q} g_q(\zeta_q^{i_q}) \leq \prod_{q=0}^{p-1} \|g_q\|_\infty < \infty.$$

Combining this with (11), (12) and again using Assumption 1, we have

$$\sup_{\varrho \in [(n+1)Mm]} |\xi_\varrho^m| \leq \frac{1}{\sqrt{Mm}} \max_{p \in \{0, \dots, n\}} \frac{\prod_{q=0}^{p-1} \|g_q\|_\infty}{\gamma_p(\mathbf{1})} \text{osc}(\overline{Q}_{p,n}(\overline{\varphi})) < \infty, \quad (13)$$

with the convention $\prod_{q=0}^{-1} \|g_q\|_\infty = 1$.

In our $m \rightarrow \infty$ analysis we consider the quantities in (11)–(12) associated with an instance of Algorithm 3 for each $m \geq 1$. We harmlessly assume that \mathbb{P} makes these instances statistically independent, but we commit an abuse, especially in (14) below, and suppress from the notation the association of $\{\zeta_p^i : i \in [Mm]\}$, and various other objects, with the particular value m .

For each $m \geq 0$ define $\mathcal{F}_0^m := \{\emptyset, \mathbb{X}\}$, and then define σ -algebras $\{\mathcal{F}_\varrho^m : 1 \leq \varrho \leq (n+1)Mm, m \geq 1\}$ recursively by

$$\mathcal{F}_\varrho^m := \mathcal{F}_{(n+1)(m-1)M}^{m-1} \vee \sigma\left(\zeta_{p_m(1)}^{i_m(1)}, \dots, \zeta_{p_m(\varrho)}^{i_m(\varrho)}\right). \quad (14)$$

With these definitions in hand, we can state the following result. The bound in (15) summarises (13), the rest of the statement is a direct application of [14, Proposition 1 and Theorem 1] and provides what we shall need: the desired martingale structure and bounds on the particle approximation errors.

Proposition 1. *Fix $n \geq 0$, $\beta > 0$ and $M \geq 1$ and suppose that Assumption (2.1) holds. For any $\varphi \in \mathcal{B}(\mathbb{X})$ there exists $C_n \in \mathbb{R}$ such that*

$$|\xi_\varrho^m| \leq \frac{1}{\sqrt{Mm}} C_n, \quad \forall m \geq 1, \varrho \in [(n+1)Mm]. \quad (15)$$

For each $m \geq 1$, $\left\{ \left(\sum_{s=1}^{\varrho} \xi_s^m, \mathcal{F}_\varrho^m \right) : \varrho \in [(n+1)Mm] \right\}$ is a zero-mean, square integrable martingale and

$$\sqrt{Mm} \Gamma_n^{Mm}(\overline{\varphi}) = \sum_{\varrho=1}^{(n+1)Mm} \xi_\varrho^m. \quad (16)$$

Moreover, for any $p \geq 1$

$$\sup_{M, m \geq 1} \sqrt{Mm} \mathbb{E}[|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)|^p]^{1/p} < \infty \quad (17)$$

$$\sup_{M, m \geq 1} \sqrt{Mm} \mathbb{E}[|\Gamma_n^{Mm}(\mathbf{1}) - \gamma_n(\mathbf{1})|^p]^{1/p} < \infty \quad (18)$$

Remark 2. By a Borel-Cantelli argument, it follows from (17)–(18) that for both the LEPF and IBPF, the particle approximation errors $\pi_n^{Mm}(\varphi) - \pi_n(\varphi)$ and $\Gamma_n^{Mm}(\mathbf{1}) - \gamma_n(\mathbf{1})$ converge to zero almost surely, both in the regime M fixed, $m \rightarrow \infty$ and in the regime m fixed, $M \rightarrow \infty$.

Remark 3. It follows from the martingale part of Proposition 1 that $\mathbb{E}[\bar{\Gamma}_n^{Mm}(\mathbf{1})] = 1$, implying that for the LEPF and IBPF, $\Gamma_n^{Mm}(\mathbf{1})$ is an unbiased approximation of the normalising constant $\gamma_n(\mathbf{1})$; a fact implying that these algorithms are also suitable for implementation as a part of a particle Markov chain Monte Carlo algorithm [17]. Some of the arguments in the proof of Theorem 1 could be adapted to establish asymptotic normality of $\sqrt{Mm}(\Gamma_n^{Mm}(\mathbf{1}) - \gamma_n(\mathbf{1}))$ with M fixed, $m \rightarrow \infty$, but the details are beyond the scope of this paper.

In order to establish asymptotic normality of $\sqrt{Mm}\bar{\Gamma}_n^{Mm}(\bar{\varphi})$ we shall apply the following special case of [16, Theorem 3.2].

Theorem 2. Fix $n \geq 0$ and $M \geq 1$. For each $m \geq 1$, suppose that $\left\{ \left(\sum_{s=1}^{\varrho} \xi_s^m, \mathcal{F}_{\varrho}^m \right) : \varrho \in [(n+1)Mm] \right\}$ is zero-mean, square integrable martingale, and that $\mathcal{F}_{\varrho}^m \subset \mathcal{F}_{\varrho}^{m+1}$ for each $\varrho \in [(n+1)Mm]$. If

$$\sum_{\varrho=1}^{(n+1)Mm} \mathbb{E} \left[(\xi_{\varrho}^m)^2 \mathbb{I}[|\xi_{\varrho}^m| > \varepsilon] \mid \mathcal{F}_{\varrho-1}^m \right] \xrightarrow[m \rightarrow \infty]{\mathbb{P}} 0, \quad \forall \varepsilon > 0, \quad (19)$$

$$\sum_{\varrho=1}^{(n+1)Mm} \mathbb{E} \left[(\xi_{\varrho}^m)^2 \mid \mathcal{F}_{\varrho-1}^m \right] \xrightarrow[m \rightarrow \infty]{\mathbb{P}} \sigma^2, \quad \sigma^2 > 0, \quad (20)$$

then

$$\sum_{\varrho=1}^{(n+1)Mm} \xi_{\varrho}^m \xrightarrow[m \rightarrow \infty]{d} \mathcal{N}(0, \sigma^2). \quad (21)$$

We now present the main arguments in the proof of Theorem 1.

Proof of Theorem 1. We first note that the case $n = 0$ is trivial, since in Algorithm 3, $\{\zeta_0^i : i \in [N]\}$ are i.i.d. samples from π_0 . So it remains to consider $n \geq 1$. With the definitions (11), (12), and (14), Proposition 1 establishes that

$$\left\{ \left(\sum_{s=1}^{\varrho} \xi_s^m, \mathcal{F}_{\varrho}^m \right) : \varrho \in [(n+1)Mm] \right\}$$

constitutes the martingale array as in the statement of Theorem 2, and our next task is to check conditions (19) and (20).

Condition (19) is easily seen to be satisfied due to (15). The majority of our work then goes into checking (20). Since, for given $m \geq 1$, $\{(\xi_{\varrho}^m, \mathcal{F}_{\varrho}^m) : \varrho \in [(n+1)Mm]\}$ is a martingale difference sequence, we have

$$\begin{aligned} \sum_{\varrho=1}^{(n+1)Mm} \mathbb{E}[(\xi_{\varrho}^m)^2 \mid \mathcal{F}_{\varrho-1}^m] &= \mathbb{E} \left[\left(\sum_{\varrho=1}^{(n+1)Mm} \xi_{\varrho}^m \right)^2 \right] \\ &\quad + \sum_{\varrho=1}^{(n+1)Mm} \mathbb{E}[(\xi_{\varrho}^m)^2 \mid \mathcal{F}_{\varrho-1}^m] - \mathbb{E}[(\xi_{\varrho}^m)^2]. \end{aligned}$$

Proposition 2 in Section 3.3 establishes convergence to zero of the residual, in the sense that

$$\sum_{\varrho=1}^{(n+1)Mm} \mathbb{E}[(\xi_{\varrho}^m)^2 \mid \mathcal{F}_{\varrho-1}^m] - \mathbb{E}[(\xi_{\varrho}^m)^2] \xrightarrow[m \rightarrow \infty]{\mathbb{P}} 0.$$

Proposition 3 in Section 3.4 establishes convergence of the variance, in the sense that

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{\varrho=1}^{(n+1)Mm} \xi_{\varrho}^m \right)^2 \right] &\xrightarrow[m \rightarrow \infty]{} \\ \frac{1}{M\gamma_n(\mathbf{1})^2} \sum_{\substack{0 \leq u < M \\ |v| \leq 2n\beta}} \mathbb{E}_{u,v+u} \left[\pi_0^{\otimes 2} \mathcal{C}_{\epsilon_0} Q_1^{\otimes 2} \mathcal{C}_{\epsilon_1} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n} (\bar{\varphi}^{\otimes 2}) \right], \end{aligned}$$

where $\epsilon_k = \mathbb{I}[I_k = J_k]$. Thus condition (20) is satisfied and so by (16) in Proposition 1,

$$\sqrt{Mm} \bar{\Gamma}_n^{Mm}(\bar{\varphi}) \xrightarrow[m \rightarrow \infty]{\text{d}} \mathcal{N}(0, \sigma_n^2). \quad (22)$$

By (2.1) we can use (17) and (18) of Proposition 1 and Hölder's inequality to

obtain

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \left[\left| \pi_n^{Mm}(\bar{\varphi}) \sqrt{Mm} \left(\Gamma_n^{Mm}(\mathbf{1}) - \gamma_n(\mathbf{1}) \right) \right| \right] \\ & \leq \lim_{m \rightarrow \infty} \mathbb{E} \left[\left| \pi_n^{Mm}(\bar{\varphi}) \right|^2 \right]^{\frac{1}{2}} \sup_{m \geq 1} \sqrt{Mm} \mathbb{E} \left[\left| \Gamma_n^{Mm}(\mathbf{1}) - \gamma_n(\mathbf{1}) \right|^2 \right]^{\frac{1}{2}} = 0, \end{aligned}$$

implying

$$\pi_n^{Mm}(\bar{\varphi}) \sqrt{Mm} \left(\Gamma_n^{Mm}(\mathbf{1}) - \gamma_n(\mathbf{1}) \right) \xrightarrow[m \rightarrow \infty]{\mathbb{P}} 0. \quad (23)$$

The claim follows by Slutsky's theorem from (10), (22) and (23). \square

3.3. Convergence of the residual to zero

Proposition 2. *Under the assumptions of Theorem 1,*

$$\sum_{\varrho=1}^{(n+1)Mm} \mathbb{E} \left[(\xi_{\varrho}^m)^2 \mid \mathcal{F}_{\varrho-1}^m \right] - \mathbb{E} \left[(\xi_{\varrho}^m)^2 \right] \xrightarrow[m \rightarrow \infty]{\mathbb{P}} 0.$$

Proof. Define:

$$Z_{\varrho}^m := \mathbb{E} \left[(\xi_{\varrho}^m)^2 \mid \mathcal{F}_{\varrho-1}^m \right] - \mathbb{E} \left[(\xi_{\varrho}^m)^2 \right].$$

By Markov's inequality we have for all $\varepsilon > 0$ that

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{\varrho=1}^{(n+1)Mm} Z_{\varrho}^m \right| \geq \varepsilon \right) & \leq \frac{1}{\varepsilon^2} \sum_{\varrho=1}^{(n+1)Mm} \mathbb{E} \left[(Z_{\varrho}^m)^2 \right] \\ & \quad + \frac{1}{\varepsilon^2} \sum_{\varrho=1}^{(n+1)Mm} \sum_{\varrho' \neq \varrho} \mathbb{E} \left[Z_{\varrho}^m Z_{\varrho'}^m \right]. \end{aligned} \quad (24)$$

By (15), $(Z_{\varrho}^m)^2 \leq 4C_n^4/(Mm)^2$ and hence

$$\sum_{\varrho=1}^{(n+1)Mm} \mathbb{E} \left[(Z_{\varrho}^m)^2 \right] \leq \frac{4(n+1)C_n^4}{Mm} \xrightarrow[m \rightarrow \infty]{} 0.$$

To establish convergence to zero of the second summation on the r.h.s. of (24), we shall show that suitably many pairs $Z_{\varrho}^m, Z_{\varrho'}^m$ are independent, therefore making no contribution to the sum since $\mathbb{E}[Z_{\varrho}^m] = 0$, and use (15) to bound the remaining pairs. Introduce the notation, for $i \in [Mm]$,

$$\text{pa}(\zeta_0^i) := \emptyset, \quad \text{pa}(\zeta_n^i) := \left\{ \zeta_q^j : 0 \leq q < n, j \in [Mm], (\alpha^{n-q})^{ij} > 0 \right\}, \quad n \geq 1, \quad (25)$$

and by convention let $\sigma(\emptyset)$ be the trivial σ -algebra. Our strategy to obtain a lower bound for the number of independent pairs $Z_\varrho^m, Z_{\varrho'}^m$ is as follows:

Lemma 1 shows that Z_ϱ^m is measurable w.r.t. $\sigma(\text{pa}(\zeta_{p_m(\varrho)}^{i_m(\varrho)}))$, and consequently

$$\sigma(\text{pa}(\zeta_{p_m(\varrho)}^{i_m(\varrho)})) \perp \sigma(\text{pa}(\zeta_{p_m(\varrho')}^{i_m(\varrho')})) \implies Z_\varrho^m \perp Z_{\varrho'}^m.$$

Lemma 2 shows that for any $0 \leq p, q \leq n$ and $i, j \in [Mm]$,

$$\text{pa}(\zeta_p^i) \cap \text{pa}(\zeta_q^j) = \emptyset \implies \sigma(\text{pa}(\zeta_p^i)) \perp \sigma(\text{pa}(\zeta_q^j)).$$

Lemma 3 shows that the number of pairs $\varrho \neq \varrho'$ such that $\text{pa}(\zeta_{p_m(\varrho)}^{i_m(\varrho)}) \cap \text{pa}(\zeta_{p_m(\varrho')}^{i_m(\varrho')}) = \emptyset$ is at least $Mm(n+1)^2(Mm - 4n\beta - 1)$.

The total number of pairs (ϱ, ϱ') where $\varrho \neq \varrho'$ is $(n+1)Mm((n+1)Mm - 1)$ and hence by (15)

$$\sum_{\varrho=1}^{(n+1)Mm} \sum_{\varrho' \neq \varrho} \mathbb{E}[Z_\varrho^m Z_{\varrho'}^m] \leq \frac{4C_n^4}{Mm} ((n+1)((n+1)Mm - 1) - (n+1)^2(Mm - 4n\beta - 1))$$

which is easily seen to converge to 0 as $m \rightarrow \infty$, completing the proof of the Proposition. \square

Before presenting Lemmata 1–3 we point out the following useful consequence of (25). Note that $\zeta_q^{i_q} \in \text{pa}(\zeta_p^{i_p})$ if and only if there exists a sequence (i_q, \dots, i_p) , such that $\prod_{k=p-1}^q \alpha^{i_{k+1}i_k} > 0$. Using this equivalence it follows that if $\ell < q < p$ and $\zeta_\ell^{i_\ell} \in \text{pa}(\zeta_q^{i_q})$ and $\zeta_q^{i_q} \in \text{pa}(\zeta_p^{i_p})$, then also $\zeta_\ell^{i_\ell} \in \text{pa}(\zeta_p^{i_p})$, and thus we have the implication

$$\zeta \in \text{pa}(\zeta_p^{i_p}) \implies \text{pa}(\zeta) \subset \text{pa}(\zeta_p^{i_p}). \quad (26)$$

Lemma 1. For any $\varrho \in [(n+1)Mm]$, Z_ϱ^m is measurable w.r.t. $\sigma(\text{pa}(\zeta_{p_m(\varrho)}^{i_m(\varrho)}))$.

Proof. The variables $\{\zeta_0^i\}_{i \in [Mm]}$ are independent, so for $\varrho \in [Mm]$, $Z_\varrho^m = 0$, \mathbb{P} -a.s. and $\sigma(\text{pa}(\zeta_{p_m(\varrho)}^{i_m(\varrho)})) = \{0, \Omega\}$, hence the claimed measurability holds. For $Mm < \varrho \leq (n+1)Mm$ we need to show that

$$\mathbb{E}[(\xi_\varrho^m)^2 \mid \mathcal{F}_{\varrho-1}^m] = \mathbb{E}[(\xi_\varrho^m)^2 \mid \text{pa}(\zeta_{p_m(\varrho)}^{i_m(\varrho)})], \quad \mathbb{P} - \text{a.s.} \quad (27)$$

According to Algorithm 3,

$$\mathbb{P}(\zeta_p^{i_p} \in A \mid \zeta_0, \dots, \zeta_{p-1}) = \frac{\sum_j \alpha^{i_p j} W_{p-1}^j g_{p-1}(\zeta_{p-1}^j) f(\zeta_{p-1}^j, A)}{W_p^{i_p}}, \quad \mathbb{P} - \text{a.s.} \quad (28)$$

Writing out the expression for $W_p^{i_p}$ from Algorithm 3 gives

$$W_p^{i_p} = \sum_{(i_0, \dots, i_{p-1})} \prod_{q=0}^{p-1} g_q(\zeta_q^{i_q}) \alpha^{i_{q+1} i_q}, \quad (29)$$

which clearly is measurable w.r.t. $\sigma(\text{pa}(\zeta_p^{i_p}))$. Noting additionally (26), we find the r.h.s. of (28) also measurable w.r.t. $\sigma(\text{pa}(\zeta_p^{i_p}))$. The latter observation combined with the fact that in Algorithm 3 the variables $\{\zeta_p^i\}_{i \in [Mm]}$ are conditionally independent given $\zeta_0, \dots, \zeta_{p-1}$, shows that

$$\mathbb{P}(\zeta_p^i \in A \mid \zeta_0, \dots, \zeta_{p-1}, \zeta_p^1, \dots, \zeta_p^{i-1}) = \mathbb{P}(\zeta_p^i \in A \mid \text{pa}(\zeta_p^i)), \quad \mathbb{P} - \text{a.s.} \quad (30)$$

Then using again the fact that W_p^i is measurable w.r.t. $\sigma(\text{pa}(\zeta_p^i))$, we have by (12) that (27) holds for $Mm < \varrho \leq Mm(n+1)$, which completes the proof. \square

Lemma 2. *For any $0 \leq p, q \leq n$ and $i, j \in [Mm]$,*

$$\text{pa}(\zeta_p^i) \cap \text{pa}(\zeta_q^j) = \emptyset \implies \sigma(\text{pa}(\zeta_p^i)) \perp \sigma(\text{pa}(\zeta_q^j)). \quad (31)$$

Proof. The implication in (31) holds immediately in the case that $p, q \in \{0, 1\}$, due to the convention that $\sigma(\emptyset)$ is the trivial σ -algebra and the independence of the ζ_0^i 's. So suppose w.l.o.g. $p > 1$ and $0 \leq q \leq p$, fix any $i, j \in [Mm]$ and assume that $\text{pa}(\zeta_p^i) \cap \text{pa}(\zeta_q^j) = \emptyset$. For $0 \leq r < p$, define the sets of random variables

$$\begin{aligned} \mathcal{Z}_r &:= \text{pa}(\zeta_p^i) \cap \{\zeta_s^k; 0 \leq s \leq r, k \in [Mm]\}, \\ \mathcal{Z}'_r &:= \text{pa}(\zeta_q^j) \cap \{\zeta_s^k; 0 \leq s \leq r, k \in [Mm]\}, \end{aligned}$$

notice that $\mathcal{Z}_{p-1} = \text{pa}(\zeta_p^i)$ and similarly $\mathcal{Z}'_{p-1} = \text{pa}(\zeta_q^j)$, so our objective is to prove $\sigma(\mathcal{Z}_{p-1}) \perp \sigma(\mathcal{Z}'_{p-1})$. Notice also that $\mathcal{Z}_r \cap \mathcal{Z}'_r = \emptyset$ for $0 \leq r < p$ since we have assumed $\text{pa}(\zeta_p^i) \cap \text{pa}(\zeta_q^j) = \emptyset$. We proceed with an inductive argument, the induction hypothesis being that for some $0 \leq r < p-1$,

$$\sigma(\mathcal{Z}_r) \perp \sigma(\mathcal{Z}'_r). \quad (32)$$

To initialise, observe that (32) holds with $r = 0$, due when $q = 0$ to the convention that $\sigma(\emptyset)$ is trivial, and due when $q > 0$ to the independence of the ζ_0^i 's and $\mathcal{Z}_0 \cap \mathcal{Z}'_0 = \emptyset$. Now assume that (32) holds for some $0 \leq r < p - 1$, for each $\zeta \in \mathcal{Z}_{r+1} \cup \mathcal{Z}'_{r+1}$ let B_ζ be an arbitrary member of \mathcal{X} and let A_ζ be the event $\{\zeta \in B_\zeta\}$. Then writing $\mathcal{G}_r := \sigma(\zeta_0, \dots, \zeta_r)$, and with the convention that products over the empty set are unity, we have

$$\begin{aligned}
& \mathbb{P} \left(\bigcap_{\zeta \in \mathcal{Z}_{r+1} \cup \mathcal{Z}'_{r+1}} A_\zeta \right) \\
&= \mathbb{E} \left[\mathbb{P} \left(\bigcap_{\zeta \in \mathcal{Z}_{r+1} \setminus \mathcal{Z}_r \cup \mathcal{Z}'_{r+1} \setminus \mathcal{Z}'_r} A_\zeta \middle| \mathcal{G}_r \right) \prod_{\zeta \in \mathcal{Z}_r \cup \mathcal{Z}'_r} \mathbb{I}[A_\zeta] \right] \\
&= \mathbb{E} \left[\mathbb{P} \left(\bigcap_{\zeta \in \mathcal{Z}_{r+1} \setminus \mathcal{Z}_r} A_\zeta \middle| \mathcal{G}_r \right) \mathbb{P} \left(\bigcap_{\zeta \in \mathcal{Z}'_{r+1} \setminus \mathcal{Z}'_r} A_\zeta \middle| \mathcal{G}_r \right) \prod_{\zeta \in \mathcal{Z}_r \cup \mathcal{Z}'_r} \mathbb{I}[A_\zeta] \right] \\
&= \mathbb{E} \left[\mathbb{P} \left(\bigcap_{\zeta \in \mathcal{Z}_{r+1} \setminus \mathcal{Z}_r} A_\zeta \middle| \sigma(\mathcal{Z}_r) \right) \mathbb{P} \left(\bigcap_{\zeta \in \mathcal{Z}'_{r+1} \setminus \mathcal{Z}'_r} A_\zeta \middle| \sigma(\mathcal{Z}'_r) \right) \prod_{\zeta \in \mathcal{Z}_r \cup \mathcal{Z}'_r} \mathbb{I}[A_\zeta] \right] \\
&= \mathbb{P} \left(\bigcap_{\zeta \in \mathcal{Z}_{r+1}} A_\zeta \right) \mathbb{P} \left(\bigcap_{\zeta \in \mathcal{Z}'_{r+1}} A_\zeta \right).
\end{aligned}$$

The first equality uses the tower property of conditional expectations and the fact that $\sigma(\mathcal{Z}_r) \vee \sigma(\mathcal{Z}'_r) \subset \mathcal{G}_r$. The second and third equalities use the following facts: in Algorithm 3, $\zeta_{r+1} = \{\zeta_{r+1}^k : k \in [Mm]\}$ are conditionally independent given \mathcal{G}_r ; for any $\zeta \in \mathcal{Z}_{r+1} \setminus \mathcal{Z}_r$ (resp. $\zeta \in \mathcal{Z}'_{r+1} \setminus \mathcal{Z}'_r$), $\mathbb{P}(A_\zeta | \mathcal{G}_r)$ is measurable w.r.t. $\sigma(\text{pa}(\zeta))$ (see (28)–(30)); $\text{pa}(\zeta) \subset \{\zeta_s^k; 0 \leq s \leq r, k \in [Mm]\}$ and by (26) $\text{pa}(\zeta) \subset \text{pa}(\zeta_p^i)$, hence $\sigma(\text{pa}(\zeta)) \subset \sigma(\mathcal{Z}_r)$ (resp. $\sigma(\text{pa}(\zeta)) \subset \sigma(\mathcal{Z}'_r)$). The fourth equality holds by the induction hypothesis. By a monotone class argument, (32) then holds with r replaced by $r + 1$, which completes the induction and hence also the proof of (31). \square

Lemma 3. *Under Assumption (2.3), the number of pairs $\varrho \neq \varrho'$ such that $\text{pa}(\zeta_{p_m}^{i_m(\varrho)}) \cap \text{pa}(\zeta_{p_m}^{i_m(\varrho')}) = \emptyset$ is at least $Mm(n+1)^2(Mm - 4n\beta - 1)$.*

Proof. We start by proving the implication

$$\prod_{q=0}^{n-1} \alpha^{i_{q+1}i_q} > 0 \implies \Delta(i_p, i_n) \leq (n-p)\beta \leq n\beta \quad \forall 0 \leq p < n. \quad (33)$$

By (2.3), $\prod_{q=0}^{n-1} \alpha^{i_{q+1}i_q} > 0$ implies $\Delta(i_{p+1}, i_p) \leq \beta, \forall 0 \leq p < n$ and then since Δ is a metric, (33) follows from the triangle inequality.

Note that by (25) and (33), $\text{pa}(\zeta_p^i) \subset \{\zeta_r^k; 0 \leq r < p, \Delta(i, k) \leq p\beta\}$ and therefore when $i = 1$, $\text{pa}(\zeta_p^i) \cap \text{pa}(\zeta_q^j) = \emptyset$ for all $0 \leq p, q \leq n$ and $j \in \{2n\beta+2, \dots, Mm-2n\beta\}$, the latter set being non-empty for all m large enough, since M, β and n are fixed. Hence for $i = 1$ fixed, there are at least $(n+1)^2(Mm-4n\beta-1)$ pairs (ζ_p^i, ζ_q^j) such that $\text{pa}(\zeta_p^i) \cap \text{pa}(\zeta_q^j) = \emptyset$. Then allowing i to vary over the set $[Mm]$ gives the lower bound as claimed. \square

3.4. Convergence of the variance

The main result of Section 3.4 is:

Proposition 3. *Under the assumptions of Theorem 1, for all $n > 0$*

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \left[\left(\sum_{\varrho=1}^{(n+1)Mm} \xi_{\varrho}^m \right)^2 \right] \\ &= \frac{1}{M\gamma_n(\mathbf{1})^2} \sum_{\substack{0 \leq u < M \\ |v| \leq 2n\beta}} \mathbb{E}_{u,v+u} \left[\pi_0^{\otimes 2} \mathcal{C}_{\epsilon_0} Q_1^{\otimes 2} \mathcal{C}_{\epsilon_1} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n} (\bar{\varphi}^{\otimes 2}) \right]. \end{aligned} \quad (34)$$

where $\epsilon_k = \mathbb{I}[I_k = J_k]$, for all $0 \leq k \leq n$.

From (16) and (9) it follows that

$$\mathbb{E} \left[\left(\sum_{\varrho=1}^{(n+1)Mm} \xi_{\varrho}^m \right)^2 \right] = \frac{Mm}{\gamma_n(\mathbf{1})^2} \mathbb{E} \left[\left(\frac{1}{Mm} \sum_i W_n^i \bar{\varphi}(\zeta_n^i) \right)^2 \right], \quad (35)$$

The first step towards proving Proposition 3 is to develop an expression for the expectation on the r.h.s. of (35) in the following Lemma, which is inspired by tensor product analysis of [18].

Lemma 4. Fix $n \in \mathbb{N}$, $M \geq 1$, $m \geq 1$ and set $N = Mm$. For any $\varphi \in \mathcal{B}(\mathbb{X})$,

$$\begin{aligned} & \mathbb{E} \left[\left(\frac{1}{N} \sum_i W_n^i \varphi(\zeta_n^i) \right)^2 \right] \\ &= \frac{1}{N^2} \sum_{(i_0:n, j_0:n)} \left(\prod_{q=0}^{n-1} \alpha^{i_{q+1}i_q} \alpha^{j_{q+1}j_q} \right) \pi_0^{\otimes 2} \mathcal{C}_{\mathbb{I}[i_0=j_0]} Q_1^{\otimes 2} \mathcal{C}_{\mathbb{I}[i_1=j_1]} \cdots Q_n^{\otimes 2} \mathcal{C}_{\mathbb{I}[i_n=j_n]} (\varphi^{\otimes 2}). \end{aligned}$$

Proof. Throughout the proof we use the shorthand notations $i_{p:q} = (i_p, \dots, i_q)$ and $j_{p:q} = (j_p, \dots, j_q)$, where $q < p$. For all $0 \leq k \leq n$, let $\mathcal{G}_k := \sigma(\zeta_0, \dots, \zeta_k)$, and let $\varphi \in \mathcal{B}(\mathbb{X}^2)$.

For all $i \in [N]$

$$\begin{aligned} & \mathbb{E} \left[(W_n^i \delta_{\zeta_n^i} \otimes W_n^i \delta_{\zeta_n^i})(\varphi) \mid \mathcal{G}_{n-1} \right] \\ &= \left(\left(\sum_{\ell} \alpha^{i\ell} W_{n-1}^{\ell} \delta_{\zeta_{n-1}^{\ell}} \right) \otimes \left(\sum_{\ell} \alpha^{i\ell} W_{n-1}^{\ell} \delta_{\zeta_{n-1}^{\ell}} \right) \right) (Q_n^{\otimes 2}(\mathcal{C}_1(\varphi))), \end{aligned}$$

and for $i \neq j$

$$\begin{aligned} & \mathbb{E} \left[(W_n^i \delta_{\zeta_n^i} \otimes W_n^j \delta_{\zeta_n^j})(\varphi) \mid \mathcal{G}_{n-1} \right] \\ &= \left(\left(\sum_{\ell} \alpha^{i\ell} W_{n-1}^{\ell} \delta_{\zeta_{n-1}^{\ell}} \right) \otimes \left(\sum_{\ell} \alpha^{j\ell} W_{n-1}^{\ell} \delta_{\zeta_{n-1}^{\ell}} \right) \right) (Q_n^{\otimes 2}(\varphi)). \end{aligned}$$

So for all $i, j \in [N]$ we have

$$\begin{aligned} & \mathbb{E} \left[(W_n^i \delta_{\zeta_n^i} \otimes W_n^j \delta_{\zeta_n^j})(\varphi) \mid \mathcal{G}_{n-1} \right] \\ &= \left(\left(\sum_{\ell} \alpha^{i\ell} W_{n-1}^{\ell} \delta_{\zeta_{n-1}^{\ell}} \right) \otimes \left(\sum_{\ell} \alpha^{j\ell} W_{n-1}^{\ell} \delta_{\zeta_{n-1}^{\ell}} \right) \right) (Q_n^{\otimes 2}(\mathcal{C}_{\mathbb{I}[i=j]}(\varphi))). \quad (36) \end{aligned}$$

In the remainder of the proof we write $\epsilon_k = \mathbb{I}[i_k = j_k]$ for brevity. From (36) we conclude that

$$\begin{aligned} & \mathbb{E} \left[\left(\left(\frac{1}{N} \sum_{i_n} W_n^{i_n} \delta_{\zeta_n^{i_n}} \right) \otimes \left(\frac{1}{N} \sum_{j_n} W_n^{j_n} \delta_{\zeta_n^{j_n}} \right) \right) (\varphi) \mid \mathcal{G}_{n-1} \right] \\ &= \frac{1}{N^2} \sum_{(i_{n-1:n}, j_{n-1:n})} \alpha^{i_n i_{n-1}} \alpha^{j_n j_{n-1}} \left(W_{n-1}^{i_{n-1}} \delta_{\zeta_{n-1}^{i_{n-1}}} \otimes W_{n-1}^{j_{n-1}} \delta_{\zeta_{n-1}^{j_{n-1}}} \right) (Q_n^{\otimes 2}(\mathcal{C}_{\epsilon_n}(\varphi))), \end{aligned}$$

which we use to initialise a backward induction. The induction assumption is

that for some $1 \leq k < n$,

$$\begin{aligned} & \mathbb{E} \left[\left(\left(\frac{1}{N} \sum_{i_n} W_n^{i_n} \delta_{\zeta_n^{i_n}} \right) \otimes \left(\frac{1}{N} \sum_{j_n} W_n^{j_n} \delta_{\zeta_n^{j_n}} \right) \right) (\varphi) \middle| \mathcal{G}_k \right] \\ &= \frac{1}{N^2} \sum_{(i_{k:n}, j_{k:n})} \left(\prod_{q=k}^{n-1} \alpha^{i_{q+1} i_q} \alpha^{j_{q+1} j_q} \right) \left(W_k^{i_k} \delta_{\zeta_k^{i_k}} \otimes W_k^{j_k} \delta_{\zeta_k^{j_k}} \right) \left(Q_{k+1}^{\otimes 2} \mathcal{C}_{\epsilon_{k+1}} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n}(\varphi) \right). \end{aligned}$$

Then applying (36) and the tower property of conditional expectations,

$$\begin{aligned} & \mathbb{E} \left[\left(\left(\frac{1}{N} \sum_i W_n^{i_n} \delta_{\zeta_n^{i_n}} \right) \otimes \left(\frac{1}{N} \sum_{j_n} W_n^{j_n} \delta_{\zeta_n^{j_n}} \right) \right) (\varphi) \middle| \mathcal{G}_{k-1} \right] \\ &= \frac{1}{N^2} \sum_{(i_{k:n}, j_{k:n})} \left(\prod_{q=k}^{n-1} \alpha^{i_{q+1} i_q} \alpha^{j_{q+1} j_q} \right) \\ & \times \mathbb{E} \left[\left(W_k^{i_k} \delta_{\zeta_k^{i_k}} \otimes W_k^{j_k} \delta_{\zeta_k^{j_k}} \right) \left(Q_{k+1}^{\otimes 2} \mathcal{C}_{\epsilon_{k+1}} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n}(\varphi) \right) \middle| \mathcal{G}_{k-1} \right] \\ &= \frac{1}{N^2} \sum_{(i_{k:n}, j_{k:n})} \left(\prod_{q=k}^{n-1} \alpha^{i_{q+1} i_q} \alpha^{j_{q+1} j_q} \right) \\ & \times \left(\left(\sum_{i_{k-1}} \alpha^{i_k i_{k-1}} W_{k-1}^{i_{k-1}} \delta_{\zeta_{k-1}^{i_{k-1}}} \right) \otimes \left(\sum_{j_{k-1}} \alpha^{j_k j_{k-1}} W_{k-1}^{j_{k-1}} \delta_{\zeta_{k-1}^{j_{k-1}}} \right) \right) \left(Q_k^{\otimes 2} \mathcal{C}_{\epsilon_k} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n}(\varphi) \right) \\ &= \frac{1}{N^2} \sum_{(i_{k-1:n}, j_{k-1:n})} \left(\prod_{q=k-1}^{n-1} \alpha^{i_{q+1} i_q} \alpha^{j_{q+1} j_q} \right) \\ & \times \left(W_{k-1}^{i_{k-1}} \delta_{\zeta_{k-1}^{i_{k-1}}} \otimes W_{k-1}^{j_{k-1}} \delta_{\zeta_{k-1}^{j_{k-1}}} \right) \left(Q_k^{\otimes 2} \mathcal{C}_{\epsilon_k} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n}(\varphi) \right), \end{aligned}$$

proving that the induction hypothesis holds at rank $k-1$. Thus

$$\begin{aligned} & \mathbb{E} \left[\left(\left(\frac{1}{N} \sum_{i_n} W_n^{i_n} \delta_{\zeta_n^{i_n}} \right) \otimes \left(\frac{1}{N} \sum_{j_n} W_n^{j_n} \delta_{\zeta_n^{j_n}} \right) \right) (\varphi) \middle| \mathcal{G}_0 \right] \\ &= \frac{1}{N^2} \sum_{(i_{0:n}, j_{0:n})} \left(\prod_{q=0}^{n-1} \alpha^{i_{q+1} i_q} \alpha^{j_{q+1} j_q} \right) \left(W_0^{i_0} \delta_{\zeta_0^{i_0}} \otimes W_0^{j_0} \delta_{\zeta_0^{j_0}} \right) \left(Q_1^{\otimes 2} \mathcal{C}_{\epsilon_1} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n}(\varphi) \right). \end{aligned}$$

Finally, since $\{\zeta_0^i : i \in [N]\}$ are i.i.d. samples from π_0 and $W_0^i = 1$ for all $i \in [N]$,

we have

$$\begin{aligned} & \mathbb{E} \left[\left(\left(\frac{1}{N} \sum_{i_n} W_n^{i_n} \delta_{\zeta_n^{i_n}} \right) \otimes \left(\frac{1}{N} \sum_{j_n} W_n^{j_n} \delta_{\zeta_n^{j_n}} \right) \right) (\varphi) \right] \\ &= \frac{1}{N^2} \sum_{(i_{0:n}, j_{0:n})} \left(\prod_{q=0}^{n-1} \alpha^{i_{q+1} i_q} \alpha^{j_{q+1} j_q} \right) \pi_0^{\otimes 2} \mathcal{C}_{\epsilon_0} Q_1^{\otimes 2} \mathcal{C}_{\epsilon_1} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n}(\varphi), \end{aligned}$$

from which the claim follows by observing that

$$\left(\left(\frac{1}{N} \sum_{i_n} W_n^{i_n} \delta_{\zeta_n^{i_n}} \right) \otimes \left(\frac{1}{N} \sum_{j_n} W_n^{j_n} \delta_{\zeta_n^{j_n}} \right) \right) (\varphi^{\otimes 2}) = \left(\frac{1}{N} \sum_{i_n} W_n^{i_n} \varphi(\zeta_n^{i_n}) \right)^2.$$

□

Proof of Proposition 3. Throughout the proof we use the shorthand notations $i_{p:q} := (i_p, \dots, i_q)$, $j_{p:q} := (j_p, \dots, j_q)$, $i_{p:q} + u := (i_p + u, \dots, i_q + u)$ and $j_{p:q} + u := (j_p + u, \dots, j_q + u)$ for any $u \in \mathbb{Z}$ and $p, q \in \mathbb{N}$ such that $q < p$. Also we define

$$\Xi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) := \pi_0^{\otimes 2} \mathcal{C}_{\mathbb{I}[i_0=j_0]} Q_1^{\otimes 2} \mathcal{C}_{\mathbb{I}[i_1=j_1]} \cdots Q_n^{\otimes 2} \mathcal{C}_{\mathbb{I}[i_n=j_n]} (\bar{\varphi}^{\otimes 2}), \quad (37)$$

and

$$\begin{aligned} \Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) &:= \prod_{q=0}^{n-1} \alpha^{i_{q+1} i_q} \alpha^{j_{q+1} j_q}, \\ \Pi_{i_n, j_n}^\infty(i_{0:n-1}, j_{0:n-1}) &:= \prod_{q=0}^{n-1} \alpha_\infty^{i_{q+1} i_q} \alpha_\infty^{j_{q+1} j_q}. \end{aligned} \quad (38)$$

By Lemma 4, we have

$$\begin{aligned} (Mm)^2 \mathbb{E} \left[\left(\frac{1}{Mm} \sum_i W_n^i \bar{\varphi}(\zeta_n^i) \right)^2 \right] \\ = \sum_{(i_{0:n}, j_{0:n})} \Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) \Xi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) \\ = A_m + B_m \end{aligned} \quad (39)$$

where A_m and B_m are obtained by partitioning the summation set:

$$\begin{aligned} A_m &:= \sum_{\substack{(i_{0:n}, j_{0:n}): \\ \Delta(i_n, j_n) > 2n\beta}} \Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) \Xi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}), \\ B_m &:= \sum_{i_n=1}^{Mm} \sum_{\substack{j_n: \\ \Delta(i_n, j_n) \leq 2n\beta}} \sum_{(i_{0:n-1}, j_{0:n-1})} \Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) \Xi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}). \end{aligned} \quad (40)$$

Note that although not explicitly shown in the notation, $\Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1})$ depends also on m through the size of matrix α , whilst $\Xi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1})$ does not.

We shall prove that $A_m = 0$ and that for all m large enough B_m is equal to the r.h.s. of (34). First consider A_m . We can use the implication (33), given in the proof of Lemma 3, and observe that if $\Delta(i_n, j_n) > 2n\beta$ and $\Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) > 0$, then by two applications of the triangle inequality

$$\Delta(i_p, j_p) \geq \Delta(i_p, j_n) - \Delta(j_p, j_n) \geq \Delta(i_n, j_n) - \Delta(i_p, i_n) - \Delta(j_p, j_n) > 0,$$

and hence $\mathbb{I}[i_p = j_p] = 0$, for all $0 \leq p \leq n$. Consequently, by using the fact that

$$\pi_0^{\otimes 2} Q_1^{\otimes 2} \cdots Q_n^{\otimes 2} (\overline{\varphi}^{\otimes 2}) = (\pi_0 Q_1 \cdots Q_n (\overline{\varphi}))^2 = \gamma_n(\mathbf{1})^2 \pi_n(\overline{\varphi})^2 = 0,$$

we have

$$A_m = \sum_{\substack{(i_{0:n}, j_{0:n}): \\ \Delta(i_n, j_n) > 2n\beta}} \Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) \pi_0^{\otimes 2} Q_1^{\otimes 2} \cdots Q_n^{\otimes 2} (\overline{\varphi}^{\otimes 2}) = 0. \quad (41)$$

Next we consider B_m . Let us start by writing

$$B_{i_n, j_n}^m := \sum_{(i_{0:n-1}, j_{0:n-1})} \Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) \Xi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}), \quad (42)$$

and

$$\phi(a_1, \dots, a_p) := ((a_1 + kM) \bmod N, \dots, (a_p + kM) \bmod N), \quad \forall (a_1, \dots, a_p) \in \mathbb{Z}^p,$$

for some fixed $k \in \mathbb{Z}$ and any $p > 0$. First we prove that B_{i_n, j_n}^m satisfies

$$B_{i_n, j_n}^m = B_{\phi(i_n, j_n)}^m. \quad (43)$$

By (2.2) we have immediately, for all $i_{0:n}, j_{0:n} \in [Mm]^{n+1}$,

$$\Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) = \Pi_{\phi(i_n, j_n)}(\phi(i_{0:n-1}, j_{0:n-1})), \quad (44)$$

and also

$$\Xi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1}) = \Xi_{\phi(i_n, j_n)}(\phi(i_{0:n-1}, j_{0:n-1})). \quad (45)$$

Combining (42), (44), (45) and using the fact that $\phi : [Mm]^n \times [Mm]^n \rightarrow$

$[Mm]^n \times [Mm]^n$ is a bijection to perform a change of variable, we can write

$$\begin{aligned}
B_{i_n, j_n}^m &= \sum_{(i_{0:n-1}, j_{0:n-1})} \Pi_{\phi(i_n, j_n)}(\phi(i_{0:n-1}, j_{0:n-1})) \Xi_{\phi(i_n, j_n)}(\phi(i_{0:n-1}, j_{0:n-1})) \\
&= \sum_{(i_{0:n-1}, j_{0:n-1})} \Pi_{\phi(i_n, j_n)}(i_{0:n-1}, j_{0:n-1}) \Xi_{\phi(i_n, j_n)}(i_{0:n-1}, j_{0:n-1}) \\
&= B_{\phi(i_n, j_n)}^m,
\end{aligned}$$

establishing (43).

Since for any $u, v \in [N]$, and $0 \leq c \leq N/2$, $\Delta(u, v) = c$ if and only if $u = (v \pm c) \overline{\text{mod } N}$, we can re-parametrise the summations in (40), and by using (43) we have

$$\begin{aligned}
B_m &= \sum_{k=0}^{m-1} \sum_{\ell=1}^M \sum_{|c| \leq 2n\beta} B_{\ell+kM, (\ell+kM+c) \overline{\text{mod } N}}^m \\
&= m \sum_{\ell=1}^M \sum_{|c| \leq 2n\beta} B_{u_0+\ell, (u_0+\ell+c) \overline{\text{mod } N}}^m
\end{aligned} \tag{46}$$

for any $0 \leq u_0 \leq (m-1)M$.

Recall (33) from the proof of Lemma 3. An analogous implication

$$\prod_{q=0}^{n-1} \alpha_{\infty}^{i_{q+1} i_q} > 0 \implies |i_p - i_n| \leq (n-p)\beta \leq n\beta \quad \forall 0 \leq p < n, \tag{47}$$

can be established for α_{∞} by using the absolute difference instead of the metric Δ .

Let us set $u_0 = 3n\beta$ and assume that $m > (u_0 + M + 3n\beta)/M$, which is legitimate since our aim is to find the limit of B_m as $m \rightarrow \infty$. We then have

$$(u_0 + \ell + c) \overline{\text{mod } N} = u_0 + \ell + c, \quad \forall \ell \in [M], |c| \leq 2n\beta, \tag{48}$$

and by using (33) and (47) one can check that when $i_n = u_0 + \ell$ and $j_n = u_0 + \ell + c$, then $\Pi_{i_n, j_n}(i_{0:n-1}, j_{0:n-1})$ and $\Pi_{i_n, j_n}^{\infty}(i_{0:n-1}, j_{0:n-1})$ are greater than zero only if $\beta < i_{q+1}$, $j_{q+1} \leq Mm - \beta$, for all $0 \leq q < n$. But by (2.4), $\alpha^{i_{q+1} i_q} = \alpha_{\infty}^{i_{q+1} i_q}$ for all $\beta < i_{q+1} \leq Mm - \beta$ and $i_q \in [Mm]$. Thus we have by

(42) and (48)

$$\begin{aligned}
& B_{u_0+\ell, (u_0+\ell+c) \bmod N}^m \\
&= \sum_{\substack{i_{0:n-1} \in \mathbb{Z}^n \\ j_{0:n-1} \in \mathbb{Z}^n}} \Pi_{u_0+\ell, u_0+\ell+c}^\infty(i_{0:n-1}, j_{0:n-1}) \Xi_{u_0+\ell, u_0+\ell+c}(i_{0:n-1}, j_{0:n-1}) \quad (49)
\end{aligned}$$

Finally we use the fact that by (2.2) and (2.4), $\alpha_\infty^{i+kM, j+kM} = \alpha_\infty^{ij}$ for all $i, j, k \in \mathbb{Z}$ and hence by (46), (49) and the fact that $\Xi_{u_0+\ell, u_0+\ell+c}(i_{0:n-1}, j_{0:n-1}) = \Xi_{\ell, \ell+c}(i_{0:n-1}, j_{0:n-1})$,

$$\begin{aligned}
\frac{B_m}{m} &= \sum_{\ell=0}^{M-1} \sum_{|c| \leq 2n\beta} \sum_{\substack{i_{0:n-1} \in \mathbb{Z}^n \\ j_{0:n-1} \in \mathbb{Z}^n}} \Pi_{\ell, \ell+c}^\infty(i_{0:n-1}, j_{0:n-1}) \Xi_{\ell, \ell+c}(i_{0:n-1}, j_{0:n-1}) \\
&= \sum_{\ell=0}^{M-1} \sum_{|c| \leq 2n\beta} \mathbb{E}_{\ell, \ell+c} \left[\pi_0^{\otimes 2} \mathcal{C}_{\epsilon_0} Q_1^{\otimes 2} \mathcal{C}_{\epsilon_1} \cdots Q_n^{\otimes 2} \mathcal{C}_{\epsilon_n} (\overline{\varphi}^{\otimes 2}) \right], \quad (50)
\end{aligned}$$

where the last form is independent of m . The claim then follows by combining (35), (39), (41) and (50). \square

4. Time-uniform convergence

Recall from Proposition 1 that for Algorithm 3, if Assumption (2.1) holds, then for each $n \in \mathbb{N}$ and $p \geq 1$,

$$\sup_{M, m \geq 1} \sqrt{Mm} \mathbb{E}[|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)|^p]^{1/p} < \infty. \quad (51)$$

In this section we establish conditions under which the LEPF and IBPF satisfy, for all $p \geq 1$:

$$\sup_{M \geq 1} \sup_{n \geq 0} \sqrt{M} \mathbb{E}[|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)|^p]^{1/p} < \infty, \quad (52)$$

and do not satisfy, for any $p \geq 1$:

$$\sup_{m \geq 1} \sup_{n \geq 0} \sqrt{m} \mathbb{E}[|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)|^p]^{1/p} < \infty, \quad (53)$$

where in (52), m is fixed and in (53), M is fixed. We note that (52) and (53) are equivalent to corresponding inequalities with $\sup_{M \geq 1}$ and $\sup_{m \geq 1}$ replaced

by $\limsup_{M \rightarrow \infty}$ and $\limsup_{m \rightarrow \infty}$ respectively, since for $\varphi \in \mathcal{B}(\mathbb{X})$, $|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)| \leq \text{osc}(\varphi) < \infty$.

We shall again leverage the fact that the LEPF and IBPF are instances of Algorithm 3, which is itself an instance of α SMC from [14], where it was shown that

$$\mathcal{E}_n^N := \frac{(\frac{1}{N} \sum_i W_n^i)^2}{\frac{1}{N} \sum_i (W_n^i)^2}, \quad N_n^{\text{eff}}(M, m) := Mm\mathcal{E}_n^{Mm},$$

play a central role in time-uniform convergence. The quantity N_n^{eff} is commonly called the *effective sample size*. Note that by Jensen's inequality we always have $\mathcal{E}_n^N \leq 1$, or equivalently $N_n^{\text{eff}}(M, m) \leq Mm$. We shall appeal to the following result, which is a special case of [14, Proposition 3] (in particular see the last displayed equation in [14, Proof of Theorem 2]).

Proposition 4. *Suppose that Assumption (2.1) holds and additionally,*

$$\exists(\delta, \epsilon) \in [1, \infty)^2 \quad \text{s.t.} \quad \sup_{n \geq 0} \sup_{x, y} \frac{g_n(x)}{g_n(y)} \leq \delta, \quad \text{and} \quad f(x, \cdot) \leq \epsilon f(y, \cdot), \quad \forall x, y \in \mathbb{X}. \quad (54)$$

Then there exists $\rho < 1$ and for each $p \geq 1$ a finite constant c_p such that for any $n \geq 0$, $M \geq 1$, $m \geq 1$ and $\varphi \in \mathcal{B}(\mathbb{X})$, Algorithm 3 has the property:

$$\mathbb{E}[|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)|^p]^{1/p} \leq \|\varphi\|_\infty \frac{c_p}{\sqrt{Mm}} \sum_{q=0}^n \rho^{n-q} \mathbb{E} \left[|\mathcal{E}_q^{Mm}|^{-p/2} \right]^{1/p}. \quad (55)$$

4.1. The regime m fixed and $M \rightarrow \infty$

We shall now show that under the assumptions of Proposition 4, both the LEPF and IBPF satisfy (52). For the LEPF, this is a new result. For the IBPF, the result is not very surprising, since it is well known that under the strong but standard hypothesis (54), a single bootstrap particle filter is time-uniformly convergent (see [2, Section 7.4.3.] and references therein). However, perhaps more surprising is the simplicity of the following argument, which applies to both the IBPF and LEPF.

It was noted in Section 2 equation (2) that for both the LEPF and IBPF, for any $k \in [m]$,

$$W_n^i = W_n^j =: \widetilde{W}_n^k, \quad \forall i, j \in G_k. \quad (56)$$

Consequently, for any $M, m \geq 1$ and $n \geq 0$,

$$\begin{aligned}
\mathcal{E}_n^{Mm} &= \frac{\left(\frac{1}{N} \sum_{k=1}^m M \widetilde{W}_n^k\right)^2}{\frac{1}{N} \sum_{k=1}^m M (\widetilde{W}_n^k)^2} \\
&= \frac{1}{m} \frac{\left(\frac{1}{\sqrt{m}} \sum_{k=1}^m \widetilde{W}_n^k\right)^2}{\frac{1}{m} \sum_{k=1}^m (\widetilde{W}_n^k)^2} \\
&= \frac{1}{m} \left(1 + \frac{\sum_{k=1}^m \sum_{\ell \neq k} \widetilde{W}_n^k \widetilde{W}_n^\ell}{\sum_{k=1}^m (\widetilde{W}_n^k)^2}\right) \geq \frac{1}{m}, \tag{57}
\end{aligned}$$

or alternatively $N_n^{\text{eff}} \geq M$. Substituting the lower bound (57) into (55) gives (52) as claimed.

4.2. The regime M fixed and $m \rightarrow \infty$

The following proposition establishes that $\limsup_{n \rightarrow \infty} \sigma_n^2 = \infty$ is a sufficient condition for failure of (53). In Sections 5.1 and 5.2 we present examples for the IBPF and LEPF such that $\lim_{n \rightarrow \infty} \sigma_n^2 = \infty$ and (54) holds.

Proposition 5. *Consider Algorithm 3. Assume that the hypotheses of Theorem 1 hold, fix $\varphi \in \mathcal{B}(\mathbb{X})$ and $M > 1$. Then for any $n \geq 0$ and $p \geq 1$,*

$$\lim_{m \rightarrow \infty} \sqrt{m} \mathbb{E}[|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)|^p]^{1/p} = \frac{\sigma_n}{\sqrt{M}} \sqrt{2} \left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p}. \tag{58}$$

If $\limsup_{n \rightarrow \infty} \sigma_n^2 = \infty$, then (53) does not hold for any $p \geq 1$. If additionally (54) holds, then for the LEPF and IBPF, for any $p \geq 1$

$$\limsup_{m \rightarrow \infty} \sup_{n \geq 0} m^{p/2} \mathbb{E} \left[\left| \sqrt{\sum_{k \in [m]} \left(\frac{\widetilde{W}_n^k}{\sum_{j \in [m]} \widetilde{W}_n^j} \right)^2} \right|^p \right] = \infty. \tag{59}$$

Remark 4. The condition in (59) clearly rules out:

$$\sup_{m \geq 1} \sup_{n \geq 0} m^p \mathbb{E} \left[\left| \max_{k \in [m]} \frac{\widetilde{W}_n^k}{\sum_{j \in [m]} \widetilde{W}_n^j} \right|^p \right] < \infty,$$

which is exactly the key hypothesis of [8] as written in (3) in the case $\epsilon = 0$. Note however, that whilst Proposition establishes that (53) does not hold, i.e. time-uniform convergence at rate $m^{-1/2}$ does not occur, we have not ruled out

the possibility that time-uniform convergence occurs at some slower rate. Moreover, our negative result is of course valid only for the specific local exchange mechanism appearing in Algorithm 1, which is only a special case of the more general framework of [8]. In Section 6.3 we shall comment on some possible algorithmic modifications to ensure time-uniform convergence.

Proof. To prove (58), we follow arguments used in the proof of [13, Theorem 12], who established a limit of the same form for a standard particle filter. We first recall the fact that for a sequence of random variables $(A_m)_{m \geq 1}$, if $A_m \xrightarrow{d} A$ for some A , and for some $p > 0$, $(|A_m|^p)_{m \geq 1}$ is uniformly integrable, then $\lim_{m \rightarrow \infty} \mathbb{E}[|A_m|^p] = \mathbb{E}[|A|^p]$, see [19, p.14, Theorem A]. As in the statement, fix $\varphi \in \mathcal{B}(\mathbb{X})$, $M > 1$ and $n \geq 0$. Then set $A_m = \sqrt{m} \pi_n^{Mm}(\varphi)$. By Theorem 1, A_m converges in distribution to a zero-mean Gaussian variable with variance σ_n^2/M . For any given $p \geq 1$ and $\delta > 0$, (51) implies $\sup_{m \geq 1} \mathbb{E}[|A_m|^{p+\delta}] < \infty$, so by [20, Lemma II.6.3], $(|A_m|^p)_{m \geq 1}$ is uniformly integrable. Therefore (58) holds.

If (53) were to hold, the r.h.s. of (58) would be upper-bounded by a finite constant possibly depending on p and M , but independent of n . The latter would contradict $\limsup_{n \rightarrow \infty} \sigma_n^2 = \infty$. Hence (53) does not hold when $\limsup_{n \rightarrow \infty} \sigma_n^2 = \infty$.

Now assume (54) holds in addition to $\limsup_{n \rightarrow \infty} \sigma_n^2 = \infty$. In order to establish (59) by a contradiction, assume that for some $p \geq 1$ there is a constant d_p such that

$$\limsup_{m \rightarrow \infty} \sup_{n \geq 0} m^{p/2} \mathbb{E} \left[\left| \sqrt{\sum_{k \in [m]} \left(\frac{\widetilde{W}_n^k}{\sum_{j \in [m]} \widetilde{W}_n^j} \right)^2} \right|^p \right] = d_p < \infty. \quad (60)$$

Since for the IBPF and LEPF, $W_n^i = W_n^j = \widetilde{W}_n^k$ for all $i, j \in G_k$, we have

$$m \sum_{k \in [m]} \left(\frac{\widetilde{W}_n^k}{\sum_{j \in [m]} \widetilde{W}_n^j} \right)^2 = 1/\varepsilon_n^{Mm}.$$

Combining this and (60) into the bound (55) of Proposition 4 gives

$$\limsup_{m \rightarrow \infty} \sup_{n \geq 0} \sqrt{m} \mathbb{E}[|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)|^p]^{1/p} \leq \|\varphi\|_\infty \frac{c_p}{\sqrt{M}} \frac{d_p}{1 - \rho} < \infty,$$

in turn implying (53), since $|\pi_n^{Mm}(\varphi) - \pi_n(\varphi)| \leq \text{osc}(\varphi) < \infty$. But we have already proved that (53) does not hold for any $p \geq 1$ when $\limsup_{n \rightarrow \infty} \sigma_n^2 = \infty$, hence the inequality in (60) does not hold for any $p \geq 1$. This completes the proof. \square

5. A closer look at the asymptotic variance

Our objective in this section is to develop more insight into the asymptotic variance in Theorem 1,

$$\sigma_n^2 = \frac{1}{M\gamma_n(\mathbf{1})^2} \sum_{\substack{0 \leq u < M \\ |v| \leq 2n\beta}} \mathbb{E}_{u,v+u} \left[\pi_0^{\otimes 2} \mathcal{C}_{\mathbb{I}[I_0=J_0]} Q_1^{\otimes 2} \mathcal{C}_{\mathbb{I}[I_1=J_1]} \cdots Q_n^{\otimes 2} \mathcal{C}_{\mathbb{I}[I_n=J_n]} (\bar{\varphi}^{\otimes 2}) \right] \quad (61)$$

for the LEPF and IBPF, especially regarding its behaviour as $n \rightarrow \infty$.

For the convenience of the reader we recall that in (61), $\mathbb{E}_{u,v}$ denotes expectation w.r.t. to the law of the bi-variate Markov chain:

$$(I_n, J_n) \sim \delta_u \otimes \delta_v, \quad (62)$$

$$\mathbb{P}(I_k = i_k, J_k = j_k \mid I_{k+1} = i_{k+1}, J_{k+1} = j_{k+1}) = \alpha_\infty^{i_{k+1}i_k} \alpha_\infty^{j_{k+1}j_k},$$

and thus the only distinction between the asymptotic variances for the LEPF and IBPF is through α_∞ , as given in (5) and (6).

To help develop insight, we consider a much simplified HMM:

$$f(x, \cdot) = \pi_0(\cdot) \quad \text{and} \quad g_n = g \in \mathcal{B}(\mathbb{X}), \quad \forall n \in \mathbb{N}. \quad (63)$$

This is obviously quite unrealistic, so let us be clear about our motives:

Firstly, (63) can be understood as being a favourable assumption for the performance of the LEPF and IBPF: $f(x, \cdot) = \pi_0(\cdot)$ implies that $\pi_n = \pi_0$ and that the particles $\{\zeta_n^i : i \in [N]\}$ in both Algorithms 1 and 2 are i.i.d. samples from π_0 for all $n \in \mathbb{N}$. Never-the-less, we shall see in Section 5.2 in conjunction with Section 4.2 that under this favourable assumption certain *negative* results can hold for the IBPF and LEPF, namely $\lim_{n \rightarrow \infty} \sigma_n^2 = \infty$ and lack of time-uniform convergence.

Secondly, we shall see that (63) makes the expression in (61) considerably more tractable, allowing us to make precise comparisons between the LEPF and IBPF. We shall see in Section 6.2 that our conclusions for this simplified HMM are consistent with results obtained by simulation for a more realistic stochastic volatility model.

Under (63), we have $\pi_n = \pi_0$, $\gamma_n(\mathbf{1}) = \pi_0 Q_{0,n}(\mathbf{1}) = \pi_0(g)^n$, and for all $\Phi \in \mathcal{B}(\mathbb{X}^2)$ and $1 \leq p \leq n$,

$$\mathcal{C}_{\mathbb{I}[I_{p-1}=J_{p-1}]} Q_p^{\otimes 2}(\Phi) = (\mathbb{I}[I_{p-1} = J_{p-1}]g^2 + \mathbb{I}[I_{p-1} \neq J_{p-1}]g^{\otimes 2})\pi_0^{\otimes 2}(\Phi)$$

so

$$\begin{aligned} & \frac{1}{\gamma_n(\mathbf{1})^2} \pi_0^{\otimes 2} \mathcal{C}_{\mathbb{I}[I_0=J_0]} Q_1^{\otimes 2} \mathcal{C}_{\mathbb{I}[I_1=J_1]} \cdots Q_n^{\otimes 2} \mathcal{C}_{\mathbb{I}[I_n=J_n]}(\bar{\varphi}^{\otimes 2}) \\ &= \mathbb{I}[I_n = J_n] \pi_0(\bar{\varphi}^2) \prod_{p=0}^{n-1} \left(1 + \mathbb{I}[I_p = J_p] \left(\frac{\pi_0(g^2)}{\pi_0(g)^2} - 1 \right) \right) \\ &= \mathbb{I}[I_n = J_n] \pi_0(\bar{\varphi}^2) (1 + c)^{Z_n}, \end{aligned} \tag{64}$$

where

$$Z_n = \sum_{p=0}^{n-1} \mathbb{I}[I_p = J_p] \tag{65}$$

and $c = \pi_0(g^2)/\pi_0(g)^2 - 1$. By (61) and (64), we thus have

$$\frac{\sigma_n^2}{\pi_0(\bar{\varphi}^2)} = \frac{1}{M} \sum_{\substack{0 \leq u < M \\ |v| \leq 2n\beta}} \mathbb{E}_{u,v+u} [\mathbb{I}[I_n = J_n] (1 + c)^{Z_n}] = \frac{1}{M} \sum_{u=0}^{M-1} \mathbb{E}_{u,u} [e^{tZ_n}] \tag{66}$$

where $t = \log(1 + c)$ and the second equality follows from the initial condition part of (62).

We thus observe the key role in the asymptotic variance played by the moment generating function of the random variable Z_n , whose interpretation is clear by (65): Z_n is the number of times the Markov chains I and J collide in n steps. Intuitively, the more frequent these collisions tend to be, the faster the growth of the asymptotic variance.

To help formalize this intuition, our next step is to characterise the law of Z_n under (62) with $u = v$, for the IBPF and the LEPF, in order to understand

how σ_n^2 behaves as $n \rightarrow \infty$. We stress that this law is a consequence only of (62) and does not depend on (63).

5.1. Law of Z_n for the IBPF

In the case of the IBPF we see immediately by inspecting α_∞ in (6) (see also Figure 2b) that when $u = v$ for any $u \in \mathbb{Z}$ in (62), I and J are sequences of i.i.d. random variables, each uniformly distributed on the set $\{\lfloor (u-1)/M \rfloor M + 1, \dots, \lfloor (u-1)/M \rfloor M + M\}$. Hence the random variables $(\mathbb{I}[I_k - J_k])_{0 \leq k < n}$ constitute a sequence of Bernoulli variables with success probability M^{-1} and consequently

$$Z_n \sim \text{Binomial}(n, 1/M), \quad (67)$$

whatever the value of u (we note that this conclusion can also be deduced from [18, Lemma 3.2], which provides a non-asymptotic variance formula for a single bootstrap particle filter, i.e. $N = M$). Hence (66) can be further simplified to

$$\frac{\sigma_n^2}{\pi_0(\bar{\varphi}^2)} = \frac{1}{M} \sum_{u=0}^{M-1} \mathbb{E}_{u,u}[e^{tZ_n}] = \mathbb{E}_{0,0}[e^{tZ_n}], \quad (68)$$

where $t = \log(1 + c)$, $c = \pi_0(g^2)/\pi_0(g)^2 - 1$.

By (67), $\mathbb{E}_{0,0}[e^{tZ_n}]$ is the moment generating function of a binomial distribution, so readily,

$$\frac{\sigma_n^2}{\pi_0(\bar{\varphi}^2)} = \left(1 + \frac{c}{M}\right)^n. \quad (69)$$

Thus when (63) holds, and assuming that $\pi_0(\bar{\varphi}^2) > 0$ and $c > 0$, for the IBPF σ_n^2 grows exponentially fast as $n \rightarrow \infty$. This can be considered a negative result for the IBPF compared to the standard bootstrap particle filter, for which it has been shown that under a variety of more realistic conditions the sequence $(\sigma_n^2)_{n \in \mathbb{N}}$ may be bounded by a finite constant, or is tight when the observation sequence is treated as random [9, 10, 11, 12, 13]. When (63) holds one can easily construct π_0 and g such that (54) holds and $c > 0$.

5.2. Example of $\sigma_n^2 \rightarrow \infty$ for the LEPF

Let us point out an example which satisfies (63), (54) and for which $\sigma_n^2 \rightarrow \infty$. Notice that for the LEPF, it follows easily from (62) and (5) that for any $u \in \mathbb{Z}$

and whatever the values of M and θ ,

$$\mathbb{E}_{u,u}[\mathbb{I}[Z_n = n]] = \mathbb{E}_{u,u} \left[\prod_{p=0}^{n-1} \mathbb{I}[I_p = J_p] \right] = \frac{1}{M^n},$$

hence we have the crude lower bound,

$$\mathbb{E}_{u,u}[e^{tZ_n}] \geq \left(\frac{\pi_0(g^2)}{\pi_0(g)^2} \frac{1}{M} \right)^n.$$

As we shall now demonstrate, one can readily construct examples for which $\frac{\pi_0(g^2)}{\pi_0(g)^2} \frac{1}{M} > 1$ and hence such that $\sigma_n^2 \rightarrow \infty$ exponentially fast for any φ with $\pi_0(\varphi^2) > 0$. Let $\mathbb{X} = \{0, 1\}$, $p \in (0, 1/M)$, $\delta \in (0, 1)$ and

$$\pi_0(0) = p, \quad \pi_0(1) = 1 - p, \quad g(0) = 1 - \delta, \quad g(1) = \delta.$$

Then, since

$$\frac{\pi_0(g^2)}{\pi_0(g)^2} = \frac{p(1-\delta)^2 + (1-p)\delta^2}{(p(1-\delta) + (1-p)\delta)^2} \xrightarrow{\delta \rightarrow 0} 1/p > M,$$

we can choose δ small enough that $\frac{\pi_0(g^2)}{\pi_0(g)^2} \frac{1}{M} > 1$, whilst satisfying $g \in \mathcal{B}(\mathbb{X})$ and $g(x) > 0$, as required for Assumption 1 and (54).

5.3. Law of Z_n for the LEPF

The interaction pattern illustrated in Figure 2a makes study of the law of Z_n more difficult for the LEPF than for the IBPF, but never-the-less we shall below derive an exact characterisation of the distribution of Z_n . Observe that Z_n depends on I and J only through the sequence of indicator variables $(\mathbb{I}[I_k = J_k])_{0 \leq k < n}$, but this sequence is unfortunately non-Markov and difficult to analyse directly. However the bi-variate process (D, E) , with $D := (D_k)_{0 \leq k \leq n}$, $E := (E_k)_{0 \leq k \leq n}$ and

$$D_k := \left\lfloor \frac{I_{n-k} - 1}{M} \right\rfloor - \left\lfloor \frac{J_{n-k} - 1}{M} \right\rfloor, \quad E_k := \mathbb{I}[I_{n-k} = J_{n-k}], \quad \forall 0 \leq k \leq n.$$

is easier to deal with.

It follows from α_∞ in (5), (62) and some elementary manipulations (omitted for brevity) that the bi-variate sequence $(D_k, E_k)_{0 \leq k \leq n}$ is Markov and for any

$u \in \mathbb{Z}$ the initial condition $(I_n, J_n) \sim \delta_u \otimes \delta_u$ implies $(D_0, E_0) \sim \delta_0 \otimes \delta_1$. Thus all statements about the law of functionals of (D, E) in the remainder of Section 5.3 hold irrespective of the particular value of $u = v$ in (62).

By similarly elementary but lengthy manipulations it can be checked that D is also Markov, with for all $1 \leq k \leq n$, $d \in \mathbb{Z}$,

$$D_k | \{D_{k-1} = d\} \sim \frac{\theta(M - \theta)}{M^2}(\delta_{d+1} + \delta_{d-1}) + \left(\frac{(M - \theta)^2}{M^2} + \frac{\theta^2}{M^2} \right) \delta_d, \quad (70)$$

and if $d_{k-1}, d_k \in \mathbb{Z}$ and $x \in \{0, 1\}$ such that $d_{k-1} = d_k = 0$, i.e. that the integer parts of $(I_{n-k} - 1)/M$ and $(J_{n-k} - 1)/M$, as well as $(I_{n-k+1} - 1)/M$ and $(J_{n-k+1} - 1)/M$, coincide, then

$$E_k | \{D_{k-1} = d_{k-1}, D_k = d_k, E_{k-1} = x\} \sim \text{Bernoulli}\left(\frac{M}{(M - \theta)^2 + \theta^2}\right) \quad (71)$$

and otherwise

$$E_k | \{D_{k-1} = d_{k-1}, D_k = d_k, E_{k-1} = x\} \sim \delta_0. \quad (72)$$

By (71) and (72), for all $b \in [n]$,

$$Z_n | \{B = b\} \sim \text{Binomial}\left(b, \frac{M}{(M - \theta)^2 + \theta^2}\right) \quad (73)$$

where

$$B := \sum_{k=1}^n \mathbb{I}[D_{k-1} = 0] \mathbb{I}[D_k = 0]. \quad (74)$$

Therefore it remains to derive the distribution of B , the distribution of Z_n is then available by marginalisation.

We will write Beta-Binomial(n, a, b) for the so-called beta binomial distribution [21] specified for any $a, b > 0$ by the probability mass function

$$p(k) = \binom{n}{k} \frac{B(k + a, n - k + b)}{B(a, b)}, \quad \forall 0 \leq k \leq n, \quad (75)$$

where $B(a, b)$ denotes the beta-function. The case Beta-Binomial($n, 1, 0$) is understood as the point mass δ_n . Moreover, we write RWZ(n) for the distribution specified by the probability mass function:

$$p(x) = \frac{2^x}{2^{2\lfloor n/2 \rfloor}} \binom{2\lfloor n/2 \rfloor - x}{\lfloor n/2 \rfloor} \quad \forall 0 \leq x \leq \lfloor n/2 \rfloor, \quad (76)$$

with convention $\binom{n}{0} = 1$ for all $n \geq 0$. As shown in [22, Theorem 2], (76) is the distribution of the number of times a symmetric simple random walk on \mathbb{Z} starting from zero returns to zero in n time-steps. The following result, in conjunction with (73), characterises the distribution of Z_n . The proof is in the Appendix.

Lemma 5. *Fix $M \geq 1$ and $\theta \in \{1, \dots, M-1\}$, let B be as defined in (74) and let \bar{B} , S and V be random variables such that*

$$\bar{B} | \{V = v, S = s\} \sim \text{Beta-Binomial}(v, s+1, n-v-s), \quad (77)$$

and

$$S | \{V = v\} \sim \text{RWZ}(n-v), \quad V \sim \text{Binomial}\left(n, \frac{(M-\theta)^2}{M^2} + \frac{\theta^2}{M^2}\right). \quad (78)$$

Then B has the same distribution as \bar{B} .

6. Interpretation of results and discussion

One of the main conclusions which can be drawn from our results thus far is quite negative: we have seen in Section 5, that for the IBPF and LEPF, the asymptotic variance can increase over time at an exponential rate. However, taken in isolation, this fact does not convey information about the relative performance of the two algorithms. The aim of Section 6 is to address this matter, qualitatively and numerically.

In Section 6.1, we continue with a toy model for which we are able to numerically evaluate asymptotic variances without simulation and explain the behaviour we see in terms of the collision count Z_n . We also examine dependence on the parameters M and θ , compare asymptotic variance values with nonasymptotic values obtained by simulation, and explore the behaviour of the effective sample size. Section 6.2 considers a more realistic stochastic volatility model, and Section 6.3 provides some concluding perspectives and describes avenues for future investigation.

6.1. Evaluation of asymptotic variances

Recall that for the toy model of Section 5, the asymptotic variances for the IBPF and LEPF are proportional to $\mathbb{E}_{0,0}[e^{tZ_n}]$, where Z_n counts collisions of the Markov chains with transition probabilities given by α_∞ . Due to the graph in Figure 2a having only one connected component, versus several in Figure 2b, it seems natural to suppose that Z_n is “typically” lower for the LEPF than for the IBPF, and thus the LEPF will exhibit lower asymptotic variance.

To explore this idea, we now use (73) and Lemma 5 to make numerical evaluations of $\mathbb{E}_{0,0}[e^{tZ_n}]$. We do so for the specific instance of the model (63) where

$$X_0 \sim \mathcal{N}(0, 1) \quad \text{and} \quad g(x) = e^{-(x+1/2)^2/2}/\sqrt{2\pi}, \quad (79)$$

and define $t_0 := \log(\pi_0(g^2)/\pi_0(g)^2) \approx .1855077$.

Figure 3a shows $\mathbb{E}_{0,0}[e^{tZ_n}]$ vs. n for the LEPF. Noting the logarithmic scale, the plot suggests that $\mathbb{E}_{0,0}[e^{tZ_n}]$ grows without bound as $n \rightarrow \infty$. In Figure 3b, R_n denotes the ratio of $\mathbb{E}_{0,0}[e^{tZ_n}]$ for the IBPF to that for the LEPF. It is apparent that R_n is growing exponentially fast with n , suggesting the interaction structure of the LEPF has significant benefits in terms of asymptotic variance.

Figure 3c compares R_n to the ratio of non-asymptotic mean square errors estimated by:

$$R_n^N := \frac{\sum_{i=1}^{N_{MC}} (\pi_{n,IBPF}^{N,i}(\varphi) - \pi_n(\varphi))^2}{\sum_{i=1}^{N_{MC}} (\pi_{n,LEPF}^{N,i}(\varphi) - \pi_n(\varphi))^2}, \quad (80)$$

where $\pi_{n,IBPF}^{N,i}(\varphi)$ and $\pi_{n,LEPF}^{N,i}(\varphi)$, $i = 1, \dots, N_{MC}$, $N_{MC} = 2000$ are independent approximations of $\pi_n(\varphi)$, with $\varphi = Id$, obtained from the IBPF and LEPF. It is apparent that as N grows, R_n^N approaches R_n and that the benefit of the LEPF over the IBPF becomes more substantial.

The main algorithmic difference between the LEPF and the IBPF is the number of particles exchanged between groups. For the IBPF, this number is 0, for the LEPF, is specified by the parameter θ . Figure 3d shows the behaviour of R_n for different values of θ . The results suggest that highest value of R_n is obtained when $\theta = M/2$, i.e. half of the particles in each group are exchanged.

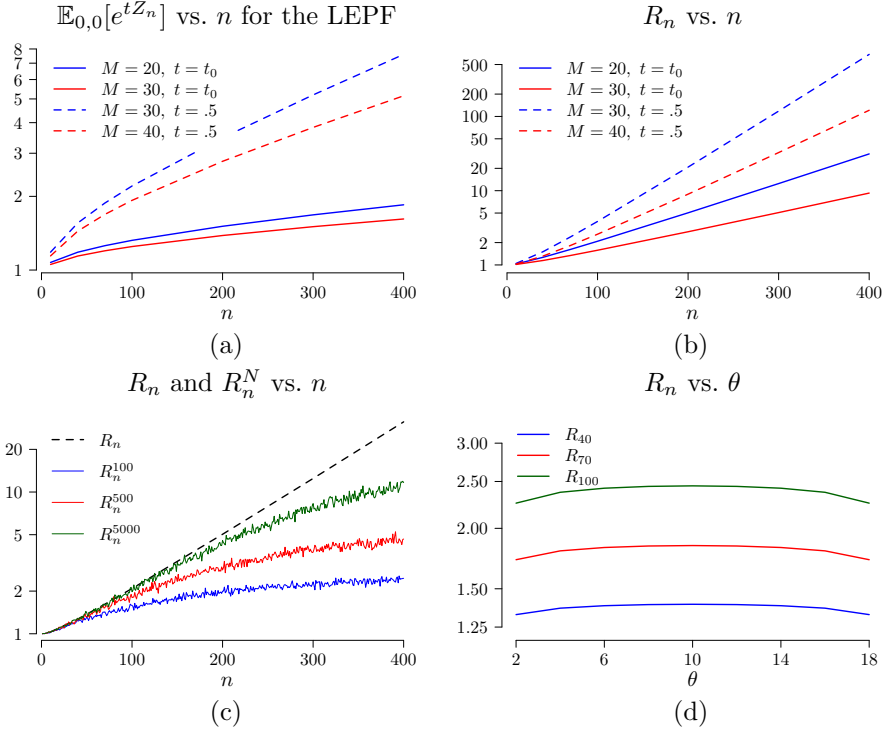


Figure 3: (a) $\mathbb{E}_{0,0}[e^{tZ_n}]$ vs. n for the LEPF with $\theta = 1$. (b) R_n vs. n for $\theta = 1$. (c) R_n and R_n^N vs. n for $\theta = 1$, $M = 20$ and $t = t_0$. (d) R_n vs. θ for $M = 20$ and $t = t_0$.

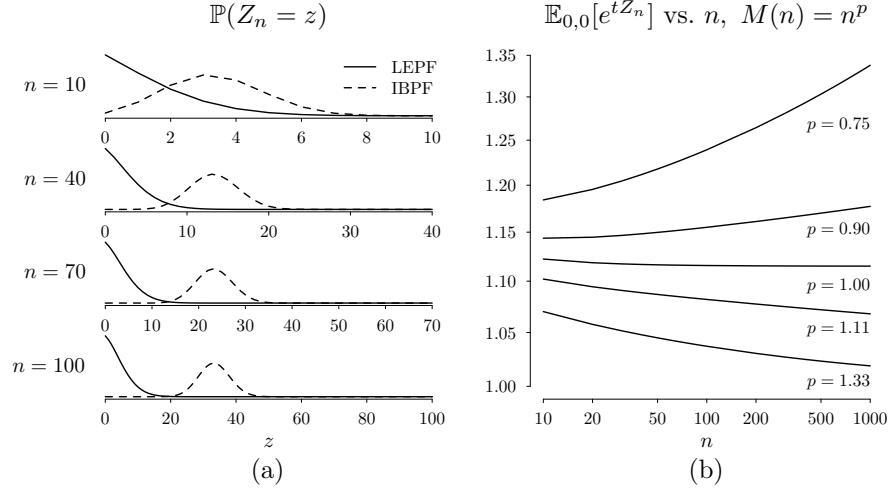


Figure 4: (a) Probability mass functions of Z_n for IBPF and LEPF with $M = 3$, $\theta = 1$. (b) $\mathbb{E}_{0,0}[e^{tZ_n}]$ vs. n for $M(n) = n^p$.

By (68), the behaviour of R_n is explained entirely by the distribution of Z_n . Figure 4a shows a comparison of these distributions in the case that $M = 3$ and $\theta = 1$, i.e. the same settings as in Figure 2. By (67) the distribution of Z_n for the IBPF is centred at n/M , while the corresponding distribution in the case of LEPF remains concentrated near 0 and, in particular, we observe that the distributions become increasingly distinct for large n .

To help illustrate the connection to the convergence results of Section 4.2, Figure 5 shows a simulation of \mathcal{E}_n^{Mm} over 70000 time steps for the LEPF using the same model as before with $M = 20$ and $m = 50, 250, 500$. For each fixed value of m , \mathcal{E}_n^{Mm} does not crash to zero and stick there, but rather it fluctuates to some extent, eventually as n grows reaching values which are closer to 0 for larger values of m . It is relevant here to recall the possibly quite loose lower bound $\mathcal{E}_n^{Mm} \geq 1/m$ derived in (57). Informally, some connection between this phenomenon and the convergence rate can be observed in equation (55), where $(\mathcal{E}_n^{Mm})^{-1/2}$ appears in the L_p error bound.

Lastly we consider the question of how $M = M(n)$ should be scaled with n

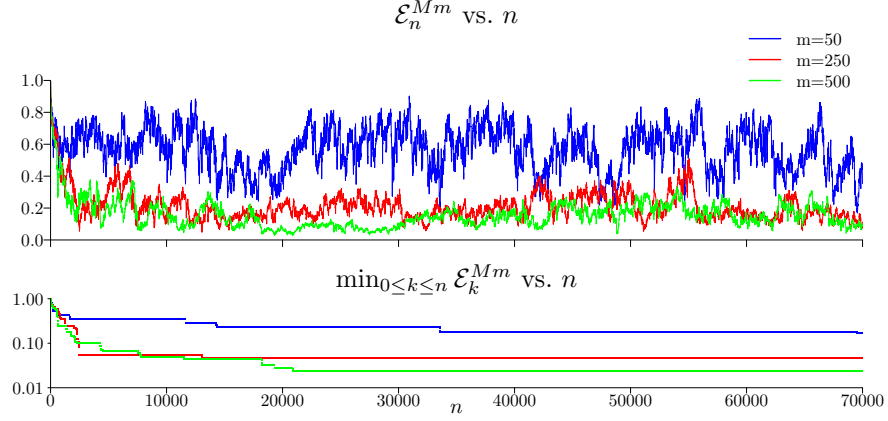


Figure 5: \mathcal{E}_n^{Mm} (top) and its running minimum on log-scale (bottom) for $M = 20$ and $m = 50, 250, 500$.

in order to prevent explosion of σ_n^2 as $n \rightarrow \infty$. So let σ_n^2 be as given in (61) but with M replaced by $M(n)$. For the IBPF we see straightforwardly that if $\limsup_{n \rightarrow \infty} n/M(n) < \infty$, then by (69),

$$\limsup_{n \rightarrow \infty} \frac{\sigma_n^2}{\pi_0(\bar{\varphi}^2)} = \limsup_{n \rightarrow \infty} \left(1 + \frac{c}{n} \frac{n}{M(n)}\right)^n < \infty.$$

We address the same issue for the LEPF through numerical evaluations again using the formulae of Section 5.3. Figure 4b shows the behaviour of $\mathbb{E}_{0,0}[e^{tZ_n}] = \sigma_n^2/\pi_0(\bar{\varphi}^2)$ for the LEPF with $M(n) = n^p$, $p = 0.75, 0.90, 1.00, 1.11, 1.33$. The results suggest that the “right” scaling may be $M(n) = n$, as for IBPF, in the sense that for $p > 1$, σ_n^2 tends towards $\pi_0(\bar{\varphi}^2)$, and for $p < 1$, $\limsup_n \sigma_n^2 = \infty$. We also note that for $M(n) = n$, we have from (79) for the IBPF that $\lim_{n \rightarrow \infty} \mathbb{E}_{0,0}[e^{tZ_n}] = e^c \approx 1.23$, where as for the LEPF in Figure 4b, with $M(n) = n$, it appears that $\limsup_{n \rightarrow \infty} \mathbb{E}_{0,0}[e^{tZ_n}] \approx 1.12$.

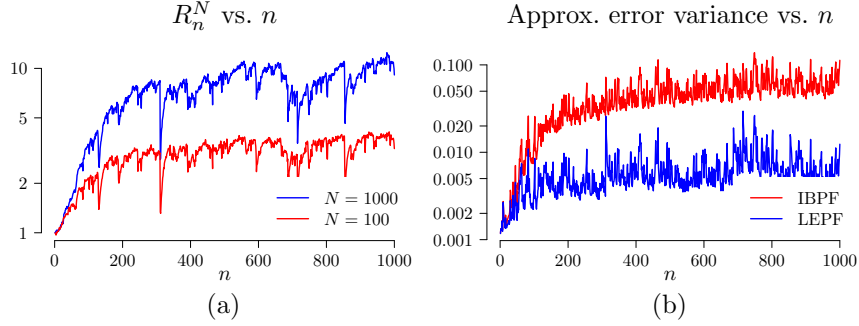


Figure 6: (a) Approximate relative variance. (b) Approximate error variance for $N = 1000$.

6.2. Simulations

We now see if some of the phenomena observed for the simplified model carry over to the case of a more realistic stochastic volatility model:

$$\begin{aligned} X_0 &\sim \mathcal{N}(0, 1), & X_{k+1} &= aX_k + V_k, & V_k &\sim \mathcal{N}(0, \sigma_V^2), & \forall k \geq 0, \\ Y_k &= b \exp(X_k/2) \varepsilon_k, & \varepsilon_k &\sim \mathcal{N}(0, 1), & \forall k \geq 0. \end{aligned} \quad (81)$$

For the parameter values in the model we took $a = 0.9$, $b = 0.1$, $\sigma_V = 0.5$, and simulated a sequence of observations from the model. For the parameters of IBPF and LEPF we took $M = 20$ and $\theta = 1$.

Figure 6a shows the ratio (80) for $N_{MC} = 10000$, $\varphi(x) = x$. The true value of $\pi_n(\varphi)$ was estimated with standard BPF using 10^6 particles. Roughly similar behaviour to that in Figure 3d can be observed, although of course for the stochastic volatility model we are not able to evaluate R_n . Figure 6b shows estimated mean square errors for IBPF and LEPF, proportional to the numerator and denominator in (80), respectively.

Figure 7 shows \mathcal{E}_n^{Mm} against n with $M = 20$ and $m = 50$ for a single run of each algorithm over 2×10^4 time steps, for the stochastic volatility model (81). For both the LEPF and the IBPF, \mathcal{E}_n^{Mm} never goes below $1/m = 0.02$, in accordance with (57), but it is notable that for the IBPF, \mathcal{E}_n^{Mm} stays quite close to $1/m = 0.02$, whereas for LEPF, \mathcal{E}_n^{Mm} fluctuates around higher values.

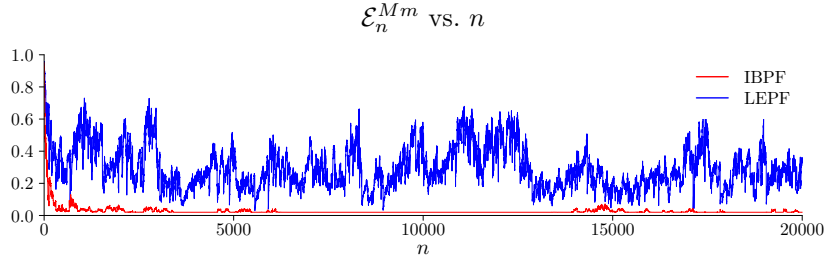


Figure 7: \mathcal{E}_n^{Mm} vs. n with $M = 20$ and $m = 50$.

6.3. Concluding remarks

Although our results establish that the asymptotic variance for the LEPF can grow over time without bound, so that time-uniform convergence at rate $m^{-1/2}$ does not hold, our numerical experiments indicate that the errors from the LEPF may be substantially smaller than those from the IBPF, and that this difference can become more substantial as the time-horizon grows. The interaction structure of the LEPF therefore has clear benefits. The question of how to maximize these benefits, by considering variants of the LEPF arising from different α matrices, seems challenging. Outside of the toy model scenario, the formula for the asymptotic variance (61) is rather complicated. However, it can be written in terms of a composition of a sequence of non-negative integral operators. If the observations (y_n) are treated as a random and stationary sequence, then the sequence of integral operators becomes also random and stationary. In light of this, Oseledec's theorem or similar results for non-negative integral operators may provide some tools to describe the rate of growth of the asymptotic variance over time.

More extreme modifications to the LEPF and IBPF may allow time-uniform convergence at rate $m^{-1/2}$ to be achieved. For instance, choosing α adaptively in a time-varying manner so as to control the effective sample size can provably help to control errors [14]. The price to pay is that doing so may compromise the

communication efficiency of the algorithm on a distributed computing architecture. Another possible approach is to stabilize the performance of the algorithm by artificially regulating the values taken by the weights W_n^i and thus introduce some bias, but avoid degeneracy and prevent low values of effective sample size. A drawback of this approach is that it would compromise the lack-of-bias properties which validate the use of particle filters within particle MCMC. Rigorous treatment of these ideas is a potential topic for future research.

7. Acknowledgement

This work was supported by the EPSRC through First Grant EP/KO23330/1 and SuSTaIn.

References

References

- [1] N. J. Gordon, D. J. Salmond, A. F. M. Smith, Novel approach to nonlinear/non-Gaussian Bayesian state estimation, *IEE Proc. F* 140 (1993) 107–113.
- [2] P. Del Moral, Feynman-Kac Formulae. Genealogical and interacting particle systems with applications, *Probab. and its Appl.*, Springer Verlag, New York, 2004.
- [3] D. Crisan, O. Obanubi, Particle filters with random resampling times, *Stoch. Proc. Appl.* 122 (2012) 1332–1368.
- [4] P. Del Moral, A. Doucet, A. Jasra, On adaptive resampling strategies for sequential Monte Carlo methods, *Bernoulli* 18 (2012) 252–278.
- [5] M. Bolić, P. M. Djurić, S. Hong, Resampling algorithms and architectures for distributed particle filters, *IEEE Trans. Sig. Process.* 53 (2005) 2442–2450.

- [6] J. Míguez, Analysis of parallelizable resampling algorithms for particle filtering, *Signal Process.* 87 (2007) 3155–3174.
- [7] J. Míguez, On the uniform asymptotic convergence of a distributed particle filter, *Proc. of the 8th Sens. Array and Multichannel Signal Process. Workshop (SAM 2014)* (2014) 241–244.
- [8] J. Míguez, M. A. Vázquez, A proof of uniform convergence over time for a distributed particle filter, *Signal Process.* 122 (2016) 152–163.
- [9] N. Chopin, Central limit theorem for sequential Monte Carlo methods and its application to Bayesian inference, *Ann. Stat.* 32 (2004) 2385–2411.
- [10] H. R. Künsch, Recursive Monte Carlo filters: algorithms and theoretical analysis, *Ann. Stat.* 33 (2005) 1983–2021.
- [11] B. Favetto, On the asymptotic variance in the central limit theorem for particle filters, *ESAIM: Probab. and Stat.* 16 (2012) 151–164.
- [12] N. Whiteley, Stability properties of some particle filters, *Ann. Appl. Probab.* 23 (2013) 2500–2537.
- [13] R. Douc, E. Moulines, J. Olsson, Long-term stability of sequential Monte Carlo methods under verifiable conditions, *Ann. Appl. Probab.* 24 (2014) 1767–1802.
- [14] N. Whiteley, A. Lee, K. Heine, On the role of interaction in sequential Monte Carlo algorithms, *Bernoulli* 22 (1) (2016) 494–529.
- [15] K. Heine, N. Whiteley, A. T. Cemgil, H. GÜltaş, Butterfly resampling: Asymptotics for particle filters with constrained interactions, *arXiv:1411.5876* (2014).
- [16] P. Hall, C. C. Heyde, *Martingale Limit Theory and Its Applications*, *Probab. and math. stat.*, Academic Press, 1980.

- [17] C. Andrieu, A. Doucet, H. Holenstein, Particle Markov chain Monte Carlo methods, J. R. Stat. Soc. Ser. B 72 (2010) 269–342.
- [18] F. Cérou, P. Del Moral, A. Guyader, A nonasymptotic theorem for un-normalized Feynman–Kac particle models, Ann. Inst. Henri Poincaré (B) Probab. Stat. 47 (2011) 629–649.
- [19] R. J. Serfling, Approximation theorems of mathematical statistics, Wiley ser. in probab. and math. stat., John Wiley & Sons, 1980.
- [20] A. N. Shiryaev, Probability, Vol. 95 of Grad. texts in math, Springer-Verlag, 1984.
- [21] J. G. Skellam, A probability distribution derived from the binomial distribution by regarding the probability of success as variable between the sets of trials, J. R. Stat. Soc. Ser. B 10 (1948) 257–261.
- [22] W. Feller, The numbers of zeros and of changes of sign in a symmetric random walk, L’Enseign. Math. 3 (1957) 229–235.
- [23] N. L. Johnson, S. Kotz, Urn models and their application, John Wiley & Sons, 1977.

Appendix A. Auxiliary proofs

Lemma 6. *The matrices defined in (5) satisfy Assumption (2.4).*

Proof. For $i, j \in \mathbb{Z}$ such that $|i - j| > \beta := M - 1 + \theta$, by (5), $\alpha_\infty^{ij} = 0$ and (4) clearly holds. For $|i - j| \leq \beta$, we observe that by (5)

$$\alpha_\infty^{ij} = \alpha_\infty^{i+kM, j+kM}, \quad \forall k \in \mathbb{Z}, \quad (\text{A.1})$$

and provided that $i + kM, j + kM \in [N]$, then by (2.2) we also have

$$\alpha_N^{i \bmod N, j \bmod N} = \alpha_N^{(i+kM) \bmod N, (j+kM) \bmod N} = \alpha_N^{i+kM, j+kM}. \quad (\text{A.2})$$

So, to complete the verification of (4) we shall, for each $i, j \in \mathbb{Z}$ such that $|i - j| \leq \beta$, find k such that $i + kM, j + kM \in [N]$ and check that $\alpha_N^{i+kM, j+kM} = \alpha_\infty^{i+kM, j+kM}$.

First consider the case $j \geq i$, and set $k = -\lfloor (i - 1)/M \rfloor$. In this case, by using (5) together with the assumptions that $Mm \geq 2\beta + 1$, and $j - i \leq \beta$ we have

$$\begin{aligned} \alpha_\infty^{i+kM, j+kM} &= M^{-1} \mathbb{I}[\lfloor (j + kM - \theta - 1)/M \rfloor = 0] \\ &= M^{-1} \mathbb{I}[j + kM - \theta \in [M]] \\ &= M^{-1} \mathbb{I}[\lfloor ((j + kM - \theta) \bmod N - 1)/M \rfloor = 0] \\ &= \alpha_N^{i+kM, j+kM}, \end{aligned}$$

and moreover $i + kM, j + kM \in [N]$ holds and thus by (A.1), (A.2) we have (4) for $j \geq i$.

For the case $i > j$, we can take $k = -\lfloor (i - 1)/M \rfloor + m - 1$, for which $i + kM, j + kM \in [N]$, and similarly as above

$$\alpha_\infty^{i+kM, j+kM} = M^{-1} \mathbb{I}[(m - 1)M \leq j + kM - \theta - 1 \leq mM - 1] = \alpha_N^{i+kM, j+kM},$$

from which we conclude, by (A.1), (A.2) that (2.4) holds for all $i, j \in \mathbb{Z}$. \square

Proof of Lemma 5. Let V be distributed as in (78), thus by (70) V has the same distribution as the number of zero increments in D . Our strategy is to construct a collection of sequences $\{D^p\}_{0 \leq p \leq V}$ and random variables $\{B^p\}_{0 \leq p \leq V}$ such that D^V and B^V have the same distributions as D and B , respectively. The construction is done in a manner that allows us to identify explicitly the distribution of B^V and hence the distribution of B .

To start, take a sequence $D^0 := (D_k^0)_{0 \leq k \leq n-V}$ where $D_0^0 = 0$ and the increments $(D_k^0 - D_{k-1}^0)_{1 \leq k \leq n-V}$ are i.i.d. with common distribution $\delta_{-1}/2 + \delta_1/2$. We then define sequences $D^p := (D_k^p)_{0 \leq k \leq n-V+p}$ for $1 \leq p \leq V$ recursively

$$D^p := (D_0^{p-1}, \dots, D_{K_p}^{p-1}, D_{K_p}^{p-1}, \dots, D_{n-V+p-1}^{p-1}), \quad (\text{A.3})$$

where K_p is a uniform random variable on the set $\{0, \dots, n - V + p - 1\}$, for all $1 \leq p \leq V$.

By this construction, D^0 is of length $n - V + 1$ and has only non-zero increments. D^1 is of length $n - V + 2$ and has exactly one zero increment at a uniformly random location. Finally, D^V is of length $n + 1$ and has exactly V zero increments at uniformly random locations and hence can be checked to have the same distribution as D .

The random variables $\{B^p\}_{0 \leq p \leq V}$ are defined as

$$B^p := \sum_{k=1}^{n-V+p} \mathbb{I}[D_{k-1}^p = 0] \mathbb{I}[D_k^p = 0], \quad \forall 0 \leq p \leq V, \quad (\text{A.4})$$

for which we have, by (A.3), the recursive expression

$$B^p = B^{p-1} + \mathbb{I}[D_{K_p}^{p-1} = 0], \quad \forall 0 < p \leq V. \quad (\text{A.5})$$

By the definition of K_p , $\mathbb{I}[D_{K_p}^{p-1} = 0]$ in (A.5) is a Bernoulli random variable with success probability

$$\frac{\sum_{k=0}^{n-V+p-1} \mathbb{I}[D_k^{p-1} = 0]}{n - V + p} = \frac{1 + B^{p-1} + \sum_{k=1}^{n-V+p-1} \mathbb{I}[D_{k-1}^{p-1} \neq 0] \mathbb{I}[D_k^{p-1} = 0]}{n - V + p}, \quad (\text{A.6})$$

and if we define $S := \sum_{k=1}^{n-V} \mathbb{I}[D_k^0 = 0]$ when $V < n$, and $S := 0$ when $V = n$. This means that S is the number of times zero occurs in the sequence D^0 , excluding the first element, then by induction, one can check that for all $0 \leq p \leq V$,

$$S = \sum_{k=1}^{n-V+p} \mathbb{I}[D_{k-1}^p \neq 0] \mathbb{I}[D_k^p = 0],$$

and hence by (A.6)

$$\mathbb{I}[D_{K_p}^{p-1} = 0] \sim \text{Bernoulli}\left(\frac{1 + S + B^{p-1}}{n - V + p}\right). \quad (\text{A.7})$$

The key observation is that by (A.5) and (A.7), the sequence $(\mathbb{I}[D_{K_p}^{p-1} = 0])_{0 < p \leq V}$, is distributed according to a Pólya's urn model, for which we have readily (see, e.g. [23])

$$\sum_{p=1}^V \mathbb{I}[D_{K_p}^{p-1} = 0] \sim \text{Beta-Binomial}(V, S + 1, n - V - S), \quad (\text{A.8})$$

with the convention that $\text{Beta-Binomial}(k, 1, 0)$ corresponds to point mass δ_k for any $k > 0$.

To conclude the proof we observe that because the increments of D^0 are non-zero, we have $B^0 = 0$ and hence, by (A.5), $B^V = \sum_{p=1}^V \mathbb{I}[D_{K_p}^{p-1} = 0]$ and therefore, by (A.8), if we set $\bar{B} = B^V$, it remains to point out that because D^0 is a simple random walk, we know by [22, Theorem 2], that S is distributed as described in (78). Finally, since D^V has the same distribution as D , then by (74) and (A.4), B^V must have the same distribution as B , concluding the proof. \square